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THE
DOCTRINE
OF
UNIVERSAL COMPARISON,
OR
GENERAL PROPORTION.

BRITISH MUSEUM

THE

DOCTRINE

OF

UNIVERSAL COMPARISON,

OR

GENERAL PROPORTION.

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DEDICATED, BY PERMISSION,

TO HIS ROYAL HIGHNESS

THE PRINCE OF WALES.

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THE following Paper, delivering geometrically the Doctrine of Univerſal Compariſon, or General Proportion, contains the geometrical inveſtigation of a theorem infinitely more general than another theorem, of which (when it is ſuppoſed to become numerical, or is applied to the algebraical values of magnitudes) the famous binomial theorem given by Sir Iſaac Newton is only a particular caſe; with a variety of other new theorems: ſhewing alſo the connection between the different abſtract ſciences, viz. Geometry, Algebra, and Arithmetic, as derivable from the ſame principles.

It is written with an intention to extend the conſideration of a new ſubject in mathematical ſcience, of which the Author has given a curſory view, in a Paper read before the Royal Society the 6th of March, 1777, and publiſhed in the Philoſophical Tranſactions; and to ſerve as an introductory paper to ſeveral ſubſequent ones, in which, amongſt other things, he purpoſes to deliver the Geometrical Inveſtigation of the Doctrine of Fluxions, Increments, and the Meaſures of Ratios, the Summation of Infinite Series geometrically, &c. &c.

THE

DOCTRINE

OF

UNIVERSAL COMPARISON, &c.

HAVING, in a Paper read before the Royal Society the 6th of March 1777, and publiſhed in the Philoſophical Tranſactions, given the demonſtration of a general geometrical formula, for finding and expreſſing a magnitude of the ſame kind with any two homogeneous magnitudes A and B, which ſhall have to B any multiplicate ratio of A to B, or a ratio compounded of the ratios of A to B, C to D, E to F, G to H, &c. ſuppoſing A and B, C and D, E and F, G and H, &c. taken two and two, to be magnitudes of the ſame, but of any kind; as alſo of a formula, for finding and expreſſing, geometrically, a magnitude of the ſame kind with any two homogeneous magnitudes A and B, which ſhall have to A any multiplicate ratio of B to A, or a ratio compounded of the ratios of B to A, C to D, E to F, G to H, &c. ſuppoſing B and A, C and D, E and F, G and H, &c. taken two and two, to be magnitudes of the ſame, but of any kind: I now intend to give the geometrical inveſtigation of general formulæ, for finding and expreſſing magnitudes of the ſame kind with any two homogeneous magnitudes A and B, having

B, having to B and A, ratios ariſing from the decompoſition of the ratios of C to D, E to F, G to H, &c. with the ratios of A to B and B to A ſeparately taken, or from the decompoſition of any multiplicate ratio of A to B, with the ſaid ratios of A to B and B to A ſeparately taken; both being neceſſary for underſtanding rightly the great variety of general theorems or formulæ contained in this paper, which I ſhall have occaſion frequently to refer to, and make uſe of, in many ſubſequent ones, that I propoſe to deliver on the following ſubjects, amongſt others, as time and leiſure from the buſineſs of an active and complicated profeſſion will permit me.

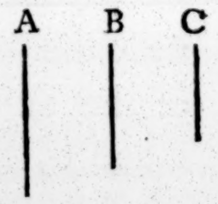
1ſt, To inveſtigate the geometrical principles of what is uſually called the doctrine of fluxions, or to deliver a method of reaſoning geometrically, applicable to every purpoſe, to which the doctrine of fluxions can be applied, without any conſideration of motion or velocity.

2dly, To inveſtigate the geometrical principles of increments, or to deliver a method of reaſoning geometrically on increaſe and decreaſe, in all the poſſible degrees of magnitude.

3dly, To inveſtigate the geometrical principles of the doctrine of the meaſures of ratios, or to deliver a method of reaſoning geometrically on the quantities of, or the various degrees of, magnitude in ratios.

4thly, To deliver a method of ſumming infinite ſeries geometrically.

5thly, To deliver the geometrical ſolutions by methods as ſtrictly ſo, as any of thoſe made uſe of in Euclid, of a great number of general problems ſimilar to the following one, which muſt lay open a new and extenſive field in ſolid geometry, and tend to unfold the great deſiderata on that ſubject, hitherto ſought for in vain, both by ancient and modern geometers.



Having any right line A whatever given; to find two cubes or ſimilar ſolids, which together ſhall have to the cube, or ſimilar ſolid, on the given line A, any ratio whatever of the ratio of any two homogeneous magnitudes B and C. Thus, for inſtance, if B be equal to C, the problem becomes this; to find geometrically two cubes, which together are equal to the cube on the given line A. If B be to C as $\sqrt{3}$ to one, and the ratio of the two cubes, or ſimilar ſolids, required to be found, to the cube or ſimilar ſolid on A, be to the ratio of B to C, as 2 is to 1, the problem becomes this; to find two cubes or ſimilar ſolids, which together are equal to thrice the cube or ſimilar ſolid on the given line A. If the ratio of the two cubes or ſimilar ſolids, required to be found, to the cube or ſimilar ſolid on A, be to the ratio of B to C, as 5 is to 1, the problem becomes this; to find two cubes or ſimilar ſolids, which together have to the cube on A, the quintuplicate ratio of B to C. In the ſame manner may examples, which are innumerable, be multiplied without end. The ſolution of this general problem, with thoſe of a variety of ſimilar, though ſomewhat different ones, reſpecting ſolid geometry, from the principles delivered in the elements of Euclid, occurred to me ſome time ago. And, although I am not ſenſible, that any geometer, either amongſt the ancients or moderns, has ſolved even one of the ſimpleſt caſes, namely, that of finding two cubes equal to any given cube, certain it is, that Euclid himſelf might have done it. Had he diſcovered the ſolution of it however, he undoubtedly would have given it. The general theorems or formulæ delivered in this paper, will greatly extend the application of ſuch ſolutions, and muſt of courſe, on this head, become the objects of future references.

I ſhall likewiſe have occaſion to refer to them frequently on a variety of other ſubjects, which it is unneceſſary at preſent to enumerate.

When I ſpeak of magnitude in this paper, I wiſh to be underſtood as taking it in its general, abſtracted, unlimited, and metaphyſical acceptation, viz. to be whatever admits of more or leſs, of increaſe or decreaſe. By quantity, I mean the degree of magnitude. When I mention the ſum of magnitudes, or ſpeak of them as additive or ſubtractive, I only mean, that they are to be taken with, or from, other magnitudes of the ſame kind. The ſigns plus and minus are only intended to denote ſuch aggregates and differences, or to connect magnitudes together in theſe relations, and by no means to convey any numerical ideas in the following theorems, or formulæ, or to imply in themſelves any ſort of myſtery, or even meaning independent of the magnitudes thus connected. By $A . \frac{C-D}{D}$, $A . \frac{A-B}{B}$, $A . \overline{\frac{A-B}{B}}\Big|^{2}$, &c. I mean a fourth proportional to D, C − D and A; a fourth proportional to B, A − B and A; a fourth proportional to B, A − B and $A . \frac{A-B}{B}$, &c. reſpectively. I am obliged to have recourſe to ſuch expreſſions in the formulæ, which ought not on any account to be conſidered as algebraic, as it is impoſſible to expreſs them by means of geometrical ſchemes or figures, in ſuch a way, as to be ſufficiently or clearly underſtood.

Geometry is the only branch of mathematical, or abſtract ſcience, that conſiders

conſiders different ſorts of dimenſions, or treats of magnitudes that are heterogeneous, or of different kinds. The diverſities however of dimenſion and magnitude, even in it, are, in reſpect of kind, only three, which we derive our ideas of from body, and the exerciſe of our external ſenſes. And as every object, with which we are ſurrounded in life, partakes of all the three, it has perhaps very fitly been called γεωμετρια, or geometry, although one of the moſt extenſive parts of it, viz. the doctrine of proportion or ratios, knows no diverſity of dimenſion or magnitude in reſpect of kind. For all ratios are homogeneous magnitudes, and differ not in kind, but in degree. How is it otherwiſe poſſible for the ratios of lines to be the ſame with, or to be equal to, the ratios of ſurfaces and ſolids, as Euclid and almoſt all other writers on geometry frequently demonſtrate, ſince no equality or ſimilitude in point of magnitude, can exiſt between things of different kinds? They never could otherwiſe ſtand to one another in the relations of greater, equal, and leſs. Neither could they ever be brought together by analogy, without ſimilitude and homogeneity. The ſimilarity of nature and homogeneity indeed of ratios, muſt always be the primary, fundamental, and leading idea in the doctrine of their meaſures. Were ratios, expreſſing the relations of lines, ſurfaces, and ſolids, to be heterogeneous, like the magnitudes themſelves, we never could reaſon from the relations of lines to thoſe of ſurfaces, nor from the relations of ſurfaces to thoſe of ſolids. And as magnitudes cannot poſſibly exiſt in any other relations to one another, in reſpect of quantity, than thoſe of greater, equal, and leſs, Euclid, after calling ſuch relations ratios, founds his definition of proportionality amongſt magnitudes, on the application of this idea to their multiples; and after defining ratios in theſe three different relations, defines analogy to be the ſimilitude of ratios. I cannot therefore help obſerving on this occaſion, that, had the learned Dr. Barrow, and the very acute and able Geometer Profeſſor Robert Simſon of Glaſgow, conſidered theſe particulars with a little more metaphyſical accuracy and attention, they would have omitted the obſervations they have made, reſpecting Euclid's third and eighth definitions on this ſubject, the one in regard to their inutility, and the other their illegitimacy. In like manner, in algebra and arithmetic, all magnitudes are homogeneous, or of the ſame kind. Thus 6^2, 6^3, 6^4, 4×6^5, &c, are all of them magnitudes of the ſame kind with 6, and with each other; and x^2, x^3, x^4, x^5, x^n, are magnitudes of the ſame kind with x, and with each other. For, otherwiſe

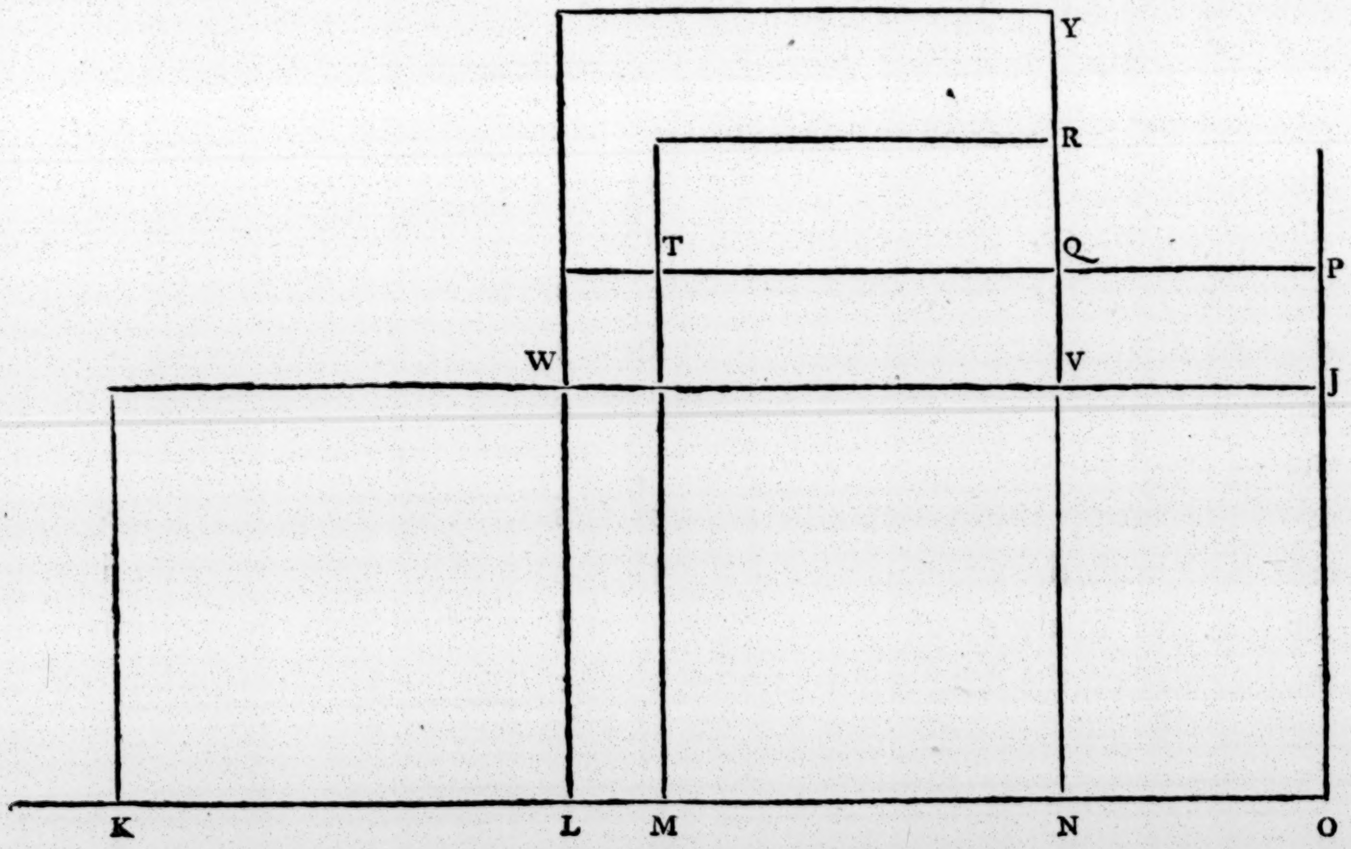
wiſe they could no more be connected together by the ſigns of addition and ſubtraction, than a line with a ſurface, or a ſurface with a ſolid. And ſuch equations as $y^4 + 32 = 4y^3 + 8y$, were not y^4, y^3 and y magnitudes of the ſame kind with each other, and with 4, 8 and 32, would imply the ſame abſurdity, as the ſuppoſition of a relation of equality between a right line and a ſurface, or between a ſurface and a ſolid. But 6^2, 6^3, 6^4, when properly conſidered, are numbers, having to 1, ratios or relations, which have to the ratio of 6 to 1, the ratios or relations of 2, 3, and 4 to 1 reſpectively; and x^2, x^3, x^4, x^5, x^n, are numbers, having to 1 ratios or relations, which have to the ratio of x to 1; the ratios of 2, 3, 4, 5 and n to 1 reſpectively. For the ideas of multiplication, diviſion, involution and evolution, are more vulgar, than accurate or ſcientific. It is likewiſe evident, that $n x^{n-1} \dot{x}$ and $\overline{n-1} \cdot x^{n-2} \dot{x}$, are only the fluxions of the numbers x^n and x^{n-1}, and are magnitudes of the ſame kind with $\dot{x}$; that the binomial theorem given by Sir Iſaac Newton, is only in reality an arithmetical one, and that $\overline{x+y}^2$, $\overline{x+y}^3$, $\overline{x+y}^4$, $\overline{x+y}^n$, are magnitudes of the ſame kind with x and y, numbers having reſpectively to 1, ratios, which have to the ratio of $x+y$ to 1, the ratios of 2, 3, 4 and n to 1. And as ratios or relations may admit of all poſſible degrees of increaſe and decreaſe, algebra and arithmetic ought to be equally well acquainted with ſuch expreſſions as $a^{\sqrt{n}}$, $b^{\sqrt[3]{m}}$, $c^{\frac{1}{\sqrt[p]{q}}}$, $d^{\frac{\sqrt[n]{r}}{\sqrt[3]{s} \times \sqrt[m]{t}}}$ &c. $6^{\sqrt{2}}$, $7^{\sqrt[3]{3}}$, $8^{\frac{\sqrt{5}}{\sqrt[3]{6}}}$, &c. and with ſuch as a^n, b^m, $c^{\frac{1}{p}}$, $d^{\frac{r}{s}}$, &c. 6^2, 7^3, $8^{\frac{5}{6}}$, &c. Geometry, then, is the only branch of abſtract ſcience which treats of heterogeneous magnitudes, or of different ſorts of dimenſions, which are three, viz. linear, ſuperficial and ſolid. Our ideas indeed of extenſion, cannot furniſh us with any other. And geometry is nothing elſe, than the application of metaphyſics to extenſion. Our reaſonings however with regard to the different degrees of quantity, in each of theſe three kinds of geometrical magnitudes, and particularly with reſpect to their properties and relations, are by no means confined to three links in the endleſs chain of univerſal compariſon, or to the ſimple duplicate and triplicate ratios, as ſeems to have been the caſe both with the ancients and moderns, but may be extended indefinitely. And as the relations of geometrical as well as of all other magnitudes, are magnitudes of the ſame kind with each other, and partake not in the leaſt of the

the dimenſions, which go to the formation of the different ſorts of extenſion and ſolidity, the general laws, which govern our reaſonings reſpecting them, muſt form the baſis, the principles, and ground-work of univerſal metricks, applicable to magnitudes of every kind, or to whatever admits of more or leſs, of increaſe or decreaſe. But geometry, when properly applied, furniſhes the inveſtigation of theſe laws or principles in the moſt unexceptionable manner, and regulates their endleſsly extenſive applications, thereby rendering all our reaſonings by means of them ſtrictly and perfectly geometrical.

Having premiſed theſe few obſervations, I now proceed to the demonſtrative part of this paper:

Let A, B; C, D; E, F; &c. be repreſented by MN, NO; OP, NR; NV, NY; &c. reſpectively; and, taken two and two, from the firſt incluſively, let them be magnitudes of the ſame, but of any kind. Let OP, NR be drawn perpendicularly to KO, or otherwiſe, if in the ſame angle; and let the rectangles or parallelograms NP, MR be completed. Let LM be a fourth proportional to OP, NR—OP and MN, and let the rectangle or parallelogram LQ be completed. Alſo let LK be a fourth proportional to NV, NY—NV and LN; and let the rectangles or parallelograms LY, KV be completed; and ſo on.

Then (14. E. 6.) LT being equal to TR, and conſequently LQ to MR, and LN having to NO (1. E. 6.) the ratio of LQ or MR to NP, LN



has to NO (23. E. 6.) the ratio, which is compounded of the ratios of MN to NO and NR to OP. — In like manner KW being equal to WY (14. E. 6.) and conſequently KV to LY; KN has to NO the ratio, which is compounded of the ratios of LN to NO and NY to NV (23. E. 6.); that is, the ratio, which is compounded of the ratios of MN to NO, NR to OP, and NY to NV. But LN is equal to MN $+ MN . \frac{NR - OP}{OP}$ or $A + A . \frac{D - C}{C}$, and KN is equal to $LN + LN . \frac{NY - NV}{NV}$ or $A + A . \frac{D - C}{C} + A . \frac{F - E}{E} + A . \frac{D - C}{C} . \frac{F - E}{E}$.—Whence it appears, that a magnitude of the ſame kind with A and B, which has to B, the ratio compounded of the ratios of A to B and D to C, is expreſſed by $A + A . \frac{D - C}{C}$; and that the magnitude which has to B, the ratio compounded of the ratios of A to B, D to C, and F to E, is expreſſed by $A + A . \frac{D - C}{C} + A . \frac{F - E}{E} + A . \frac{D - C}{C} . \frac{F - E}{E}$; and ſo on.—Theſe inferences may likewiſe be eaſily drawn from the above mentioned paper publiſhed in the Tranſactions. Now it is clear, that the ratio of D to C, compounded with, or taken together with, the ratio of A to B, produces the ſame ratio, as the ratio of C to D, decompounded with, or its inverſe, taken together with the ratio of A to B. Hence it is manifeſt, that if $(p - 1)$ denote the number of the ratios decompounded with the ratio of A to B, either of the two expreſſions $A, - A . \frac{C - D}{C} -$ &c. to $(\frac{p-1}{1})$ terms, $+ A . \frac{C - D}{C} . \frac{E - F}{E} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, $- +$ &c. or $A, + A . \frac{D - C}{C} +$ &c. to $(\frac{p-1}{1})$ terms, $+ A . \frac{D - C}{C} . \frac{F - E}{E} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms + &c. &c. has to B, a ratio, which is produced by decompounding the ratio of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratios of A to B; or by taking the inverſe of the ſame together with the ratio of A to B. If the ratios of C to D, E to F, G to H, &c, be reſpectively equal to the ratio of A to B, theſe expreſſions will become the following ones:

$$\left.\begin{array}{l} A + \frac{p-1}{1} . A . \frac{B-A}{A} + \frac{p-1}{1} . \frac{p-2}{2} . A . \left.\overline{\frac{B-A}{A}}\right|^{2} + \frac{p-1}{1} . \frac{p-2}{2} . \frac{p-3}{3} . A . \left.\overline{\frac{B-A}{A}}\right|^{3} + \&c. \\ \text{or } \frac{p-1}{1} . B - \frac{p-2}{1} . A + \frac{p-1}{1} . \frac{p-2}{2} . \frac{\overline{B-A}^{2}}{A} + \frac{p-1}{1} . \frac{p-2}{2} . \frac{p-3}{3} . \frac{\overline{B-A}^{3}}{A^{2}} + \&c. \end{array}\right\}$$

and

$$\left.\begin{array}{l}\text{and } A - \frac{p-1}{1}.\overline{A-B} + \frac{p-1}{1}.\frac{p-2}{2}.\frac{\overline{A-B}^2}{A} - \frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}.\frac{\overline{A-B}^3}{A^2} + - \text{\&c. \&c.}\\ \text{or } \frac{p-1}{1}.B - \frac{p-2}{1}.A + \frac{p-1}{1}.\frac{p-2}{2}.\frac{\overline{A-B}^2}{A} - \frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}.\frac{\overline{A-B}^3}{A^2} + - \text{\&c. \&c.}\end{array}\right\}$$

$$\text{or } B + \frac{p-2}{1}.\overline{B-A} + \frac{p-1}{1}.\frac{p-2}{2}.\frac{\overline{B-A}^2}{A} + \frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}.\frac{\overline{B-A}^3}{A^2} + \text{\&c.}$$

$$\text{and } B - \frac{p-2}{1}.\overline{A-B} + \frac{p-1}{1}.\frac{p-2}{2}.\frac{\overline{A-B}^2}{A} - \frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}.\frac{\overline{A-B}^3}{A^2} + - \text{\&c. \&c.}$$

All which expreſſions terminate—but if the ratios of C to D, E to F, G to H, &c. be not, ſeparately taken, equal to the ratio of A to B, but have to it reſpectively the ratios of *i* to *k*, *l* to *m*, *n* to *o*, &c. and *v* be ſuppoſed equal to $\frac{i}{k} + \frac{l}{m} + \frac{n}{o} +$ &c. we manifeſtly get the two expreſſions

$$B + \frac{v-1}{1}.B.\frac{B-A}{A} + \frac{v-1}{1}.\frac{v-2}{2}.B.\overline{\frac{B-A}{A}}\Big|^2 + \frac{v-1}{1}.\frac{v-2}{2}.\frac{v-3}{3}.\overline{\frac{B-A}{A}}\Big|^3 + \text{\&c.}$$

$$A + \frac{v}{1}.A.\frac{B-A}{A} + \frac{v}{1}.\frac{v-1}{2}.A.\overline{\frac{B-A}{A}}\Big|^2 + \frac{v}{1}.\frac{v-1}{2}.\frac{v-2}{3}.A.\overline{\frac{B-A}{A}}\Big|^3 + \text{\&c.}$$

having [illegible] to B, the ratio which ariſes from the decompoſition of the ratios C to D, E to F, G to H, &c. with the ratio of A to B. Now it is evident that, when *v* is a whole number, or the ratio expreſſed is any multiplicate ratio, theſe expreſſions always terminate; and that, when the ratio expreſſed is any ſub-multiplicate ratio, or when *v* is any fraction, ſuppoſing the ratios of *i* to *k*, *l* to *m*, *n* to *o*, &c. to denote relations of any pairs of homogeneous magnitudes whatever, in point of quantity, they do not terminate, but run on into indefinite or infinite geometrical ſeries. Their equality, however, to the expreſſion A, $+ A.\frac{D-C}{C} +$ &c. to $(\frac{p-1}{1})$ terms, $+ A.\frac{D-C}{C}.\frac{F-E}{E} +$ &c. to $(\frac{p-1}{1}.\frac{p-2}{2})$ terms, + &c. &c. continues both when they do and do not terminate. This might be ſhewn geometrically, but, as the proceſs at full length would ſwell this paper beyond its intended ſize, I omit ſetting it down at preſent.

The ſame obſervations hold (in the paper referred to above, as publiſhed in the Philoſophical Tranſactions) with regard to the expreſſion A, $+ A.\frac{C-D}{D} + A.\frac{E-F}{F} +$ &c. to $(\frac{p-1}{1})$ terms, $+ A.\frac{C-D}{D}.\frac{E-F}{F} +$ &c. to $(\frac{p-1}{1}.\frac{p-2}{2})$ terms, $+ A.\frac{C-D}{D}.\frac{E-F}{F}.\frac{G-H}{H} +$ &c. to $(\frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3})$ terms, + &c. &c. having to B a ratio, which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $\frac{p-1}{1}$, with the

the ratio of A to B. For, ſuppoſing the ratios of C to D, E to F, G to H, &c. to have reſpectively to the ratio of A to B, the ratios of i to k, l to m, n to o, &c. and r equal to $1 + \frac{i}{k} + \frac{l}{m} + \frac{n}{o} +$ &c. it becomes

$$\left.\begin{aligned}&A + \frac{r-1}{1}.A.\frac{A-B}{B} + \frac{r-1}{1}.\frac{r-2}{2}.A.\overline{\frac{A-B}{B}}\Big|^{2} + \frac{r-1}{1}.\frac{r-2}{2}.\frac{r-3}{3}.A.\overline{\frac{A-B}{B}}\Big|^{3} + \&c.\\ &\text{or } B + \frac{r}{1}.B.\frac{A-B}{B} + \frac{r}{1}.\frac{r-1}{2}.B.\overline{\frac{A-B}{B}}\Big|^{2} + \frac{r}{1}.\frac{r-1}{2}.\frac{r-2}{3}.B.\overline{\frac{A-B}{B}}\Big|^{3} + \&c.\end{aligned}\right\}$$

which expreſſions do or do not terminate on the ſame conditions as the foregoing ones for the decompoſition of ratios.

As the degrees of magnitude in ratios, as well as in the magnitudes themſelves, of which they expreſs the relations, are endleſs and innumerable, I deliver the following theorems in a general form, applicable to all poſſible caſes, either when the formulæ terminate or do not terminate, when they repreſent multiplicate or ſub-multiplicate ratios, or in ſhort ratios, which are to each other as any homogeneous magnitudes whatever.

THEOREM I.

Let A, B; C, D; E, F; G, H; &c. taken two and two, from the firſt incluſively, be magnitudes of the ſame, but of any kind—Then

$$\left.\begin{aligned}&\text{1ſt, } A, + A.\frac{C-D}{D} + A.\frac{E-F}{F} + \&c. \text{ to } \left(\frac{p-1}{1}\right) \text{ terms, } + A.\frac{C-D}{D}.\frac{E-F}{F} + \&c. \text{ to } \left(\frac{p-1}{1}.\frac{p-2}{2}\right) \text{ terms, } + \&c. \&c.\\ &\text{2d, Or } B, + B.\frac{A-B}{B} + B.\frac{C-D}{D} + \&c. \text{ to } \left(\frac{p}{1}\right) \text{ terms, } + B.\frac{A-B}{B}.\frac{C-D}{D} + \&c. \text{ to } \left(\frac{p}{1}.\frac{p-1}{2}\right) \text{ terms, } + \&c. \&c.\end{aligned}\right\}$$

has to B, the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $\left(\frac{p-1}{1}\right)$, with the ratio of A to B.

And

$$\left.\begin{aligned}&\text{3d, } B, + B.\frac{C-D}{D} + B.\frac{E-F}{F} + \&c. \text{ to } \left(\frac{p-1}{1}\right) \text{ terms, } + B.\frac{C-D}{D}.\frac{E-F}{F} + \&c. \text{ to } \left(\frac{p-1}{1}.\frac{p-2}{2}\right) \text{ terms, } + \&c. \&c.\\ &\text{4th, Or } A, + A.\frac{B-A}{A} + A.\frac{C-D}{D} + \&c. \text{ to } \left(\frac{p}{1}\right) \text{ terms, } + A.\frac{B-A}{A}.\frac{C-D}{D} + \&c. \text{ to } \left(\frac{p}{1}.\frac{p-1}{2}\right) \text{ terms, } + \&c. \&c.\end{aligned}\right\}$$

has to A the ratio, which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $\left(\frac{p-1}{1}\right)$ with the ratio of B to A.

The ſecond formula in this theorem is derived from the firſt, by ſubſtituting $B + B.\frac{A-B}{B}$ for A; and the fourth formula is derived from the third, by ſubſtituting $A + A.\frac{B-A}{A}$ for B.

Cor. 1. A has to A, $+ A.\frac{C-D}{D} + A.\frac{E-F}{F} +$ &c. to $\left(\frac{p-1}{1}\right)$ terms, $+ A.\frac{C-D}{D}.\frac{E-F}{F} +$ &c. to $\left(\frac{p-1}{1}.\frac{p-2}{2}\right)$ terms, + &c. &c. the ſame ratio, that

$B, + B . \frac{D-C}{C} + B . \frac{F-E}{E} +$ &c. to $(\frac{p-1}{1})$ terms, $+ B . \frac{D-C}{C} . \frac{F-E}{E} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. has to B.

Or 1 has to 1, $+ \frac{C-D}{D} + \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1})$ terms, $+ \frac{C-D}{D} . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. the ſame ratio, that 1, $+ \frac{D-C}{C} + \frac{F-E}{E} +$ &c. to $(\frac{p-1}{1})$ terms, $+ \frac{D-C}{C} . \frac{F-E}{E} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. has to 1.

Cor. 2. $A, + A . \frac{A-B}{B} + A . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{A-B}{B} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. has to A, the ſame ratio with that, which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of A to B.

Alſo $B, + B . \frac{B-A}{A} + A . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ B . \frac{B-A}{A} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. has to B, the ſame ratio with that, which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of B to A.

Cor. 3. If in the expreſſion $B + B . \frac{A-B}{B} = A$, we ſubſtitute any other magnitude M for B, we evidently get the following expreſſion, $M, + M . \frac{A-M}{M} + M . \frac{C-D}{D} + M . \frac{E-F}{F} +$ &c. to $(\frac{p}{1})$ terms, $+ M . \frac{A-M}{M} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. having to B, the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of A to B.

Alſo, if in the expreſſion $A + A . \frac{B-A}{A} = B$, we ſubſtitute any other magnitude N for A, we evidently get the following expreſſion, $N, + N . \frac{B-N}{N} + N . \frac{C-D}{D} + N . \frac{E-F}{F} +$ &c. to $(\frac{p}{1})$ terms, $+ N . \frac{B-N}{N} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{1})$ terms, + &c. &c. having to A, the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of B to A.

Cor. 4.

Cor. 4. A has to A, $+ A . \frac{B-A}{A} + A . \frac{D-C}{C}$ + &c. to $(\frac{p}{1})$ terms, $+ A . \frac{B-A}{A} . \frac{D-C}{C}$ + &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. the ſame ratio, that A, $+ A . \frac{A-B}{B} + A . \frac{C-D}{D}$ + &c. to $(\frac{p}{1})$ terms, $+ A . \frac{A-B}{B} . \frac{C-D}{D}$ + &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. has to A, or the ratio which A, $+ A . \frac{C-D}{D} + A . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1})$ terms, $+ A . \frac{C-D}{D} . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. has to B.

Alſo B has to B, $+ B . \frac{A-B}{B} + B . \frac{C-D}{D}$ + &c. to $(\frac{p}{1})$ terms, $+ B . \frac{A-B}{B} . \frac{C-D}{D}$ + &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. the ratio, that $B + B . \frac{B-A}{A} + B . \frac{D-C}{C}$ + &c. to $(\frac{p}{1})$ terms, $+ B . \frac{B-A}{A} . \frac{D-C}{C}$ + &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. has to B; or B has to B, $+ B . \frac{B-A}{A} + B . \frac{D-C}{C}$ + &c. to $(\frac{p}{1})$ terms, $+ B . \frac{B-A}{A} . \frac{D-C}{C}$ + &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. the ratio, that B, $+ B . \frac{C-D}{D} + B . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1})$ terms, $+ B . \frac{C-D}{D} . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. has to A.

Cor. 5. B has to B, $+ B . \frac{D-C}{C} + B . \frac{F-E}{E}$ + &c. to $(\frac{p-1}{1})$ terms, $+ B . \frac{D-C}{C} . \frac{F-E}{E}$ + &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. the ratio, that B, $+ B . \frac{A-B}{B} + B . \frac{C-D}{D} + B . \frac{E-F}{F}$ + &c. to $(\frac{p}{1})$ terms, $+ B . \frac{A-B}{B} . \frac{C-D}{D}$ + &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. or A, $+ A . \frac{C-D}{D} + A . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1})$ terms, $+ A . \frac{C-D}{D} . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. has to A. Or 1 has to 1, $+ \frac{D-C}{C} + \frac{F-E}{E}$ + &c. to $(\frac{p-1}{1})$ terms, $+ \frac{D-C}{C} . \frac{F-E}{E}$ + &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. the ratio, that 1, $+ \frac{C-D}{D} + \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1})$ terms, $+ \frac{C-D}{D} . \frac{E-F}{F}$ + &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c. has to 1. This follows likewiſe from Cor. 1.

Cor. 6. Hence it is manifeſt, that the ſquare on A is equal to the rectangle contained by A, $+ A . \frac{B-A}{A} + A . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. and A, $+ A . \frac{A-B}{B} + A . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{A-B}{B} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. that the ſquare on B is equal to the rectangle contained by B, $+ B . \frac{A-B}{B} + B . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ B . \frac{A-B}{B} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. and B, $+ B . \frac{B-A}{A} + B . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ B . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c. that the rectangle contained by A and B is equal to the rectangle contained

by A, $+ A . \frac{C-D}{D} + A . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1})$ terms, $+ A . \frac{C-D}{D} . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c.
and B, $+ B . \frac{D-C}{C} + B . \frac{F-E}{E} +$ &c. to $(\frac{p-1}{1})$ terms, $+ B . \frac{D-C}{C} . \frac{F-E}{E} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c.
or by A, $+ A . \frac{B-A}{A} + A . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
and A, $+ A . \frac{C-D}{D} + A . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1})$ terms, $+ A . \frac{C-D}{D} . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c.
or by B, $+ B . \frac{B-A}{A} + B . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ B . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
and B, $+ B . \frac{C-D}{D} + B . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1})$ terms, $+ B . \frac{C-D}{D} . \frac{E-F}{F} +$ &c. to $(\frac{p-1}{1} . \frac{p-2}{2})$ terms, + &c. &c.

Cor. 7. It is alſo manifeſt, that the rectangle contained by A and
$A . \frac{B-A}{A} + A . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{1})$ terms, + &c. &c.
together with the rectangle contained by A and
$A . \frac{A-B}{B} + A . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{A-B}{B} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
together with the rectangle contained by
$A . \frac{B-A}{A} + A . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
and $A . \frac{A-B}{B} + A . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ A . \frac{A-B}{B} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
vaniſh, or, taken together, are equal to (*o*); and that the rectangle contained by B and
$B . \frac{A-B}{B} + B . \frac{C-D}{D} +$ &c. to $(\frac{p}{1})$ terms, $+ B . \frac{A-B}{B} . \frac{C-D}{D} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
together with the rectangle contained by B and
$B . \frac{B-A}{A} + B . \frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ B . \frac{B-A}{A} . \frac{D-C}{C} +$ &c. to $(\frac{p}{1} . \frac{p-1}{2})$ terms, + &c. &c.
together with the rectangle contained by

$$B\cdot\frac{A-B}{B}+B\cdot\frac{C-D}{D}+\text{\&c. to }\left(\frac{p}{1}\right)\text{ terms, }+B\cdot\frac{A-B}{B}\cdot\frac{C-D}{D}+\text{\&c. to }\left(\frac{p}{1}\cdot\frac{p-1}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

$$\text{and }B\cdot\frac{B-A}{A}+B\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\right)\text{ terms, }+B\cdot\frac{B-A}{A}\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\cdot\frac{p-1}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

likewiſe vaniſh, or, taken together, are equal to (0);

and that the rectangle contained by B and

$$A\cdot\frac{C-D}{D}+A\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+A\cdot\frac{C-D}{D}\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

together with the rectangle contained by A and

$$B\cdot\frac{D-C}{C}+B\cdot\frac{F-E}{E}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+B\cdot\frac{D-C}{C}\cdot\frac{F-E}{E}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

together with the rectangle contained by

$$A\cdot\frac{C-D}{D}+A\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+A\cdot\frac{C-D}{D}\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

$$\text{and }B\cdot\frac{D-C}{C}+B\cdot\frac{F-E}{E}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+B\cdot\frac{D-C}{C}\cdot\frac{F-E}{E}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

likewiſe vaniſh, or, taken together, are equal to (0).

It is alſo evident, that the rectangle contained by A and

$$A\cdot\frac{B-A}{A}+A\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\right)\text{ terms, }+A\cdot\frac{B-A}{A}\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\cdot\frac{p-1}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

together with the rectangle contained by A and

$$A\cdot\frac{C-D}{D}+A\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+A\cdot\frac{C-D}{D}\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

together with the rectangle contained by

$$A\cdot\frac{B-A}{A}+A\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\right)\text{ terms, }+A\cdot\frac{B-A}{A}\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\cdot\frac{p-1}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

$$\text{and }A\cdot\frac{C-D}{D}+A\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+A\cdot\frac{C-D}{D}\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

are equal to the difference between the rectangle contained by B and A and the ſquare on A; and when B is greater than A, are together an additive magnitude; when B is leſs than A, a ſubtractive magnitude; but when B is equal to A, vaniſh; and that the rectangle contained by B and

$$B\cdot\frac{B-A}{A}+B\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\right)\text{ terms, }+B\cdot\frac{B-A}{A}\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\cdot\frac{p-1}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

together with the rectangle contained by B and

$$B\cdot\frac{C-D}{D}+B\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+B\cdot\frac{C-D}{D}\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

together with the rectangle contained by

$$B\cdot\frac{B-A}{A}+B\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\right)\text{ terms, }+B\cdot\frac{B-A}{A}\cdot\frac{D-C}{C}+\text{\&c. to }\left(\frac{p}{1}\cdot\frac{p-1}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

$$\text{and }B\cdot\frac{C-D}{D}+B\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\right)\text{ terms, }+B\cdot\frac{C-D}{D}\cdot\frac{E-F}{F}+\text{\&c. to }\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\right)\text{ terms, }+\text{\&c. \&c.}$$

are equal to the difference between the rectangle contained by B and A and the ſquare on B; and when B is greater than A, are together a ſubtractive magnitude; when B is leſs than A, an additive magnitude; and when B is equal to A, vaniſh.

Theſe Corollaries, with the variety of inferences deducible from them, (but which I purpoſely at preſent omit) will be found of the greateſt uſe and advantage, in many geometrical inveſtigations, and in the ſummation of ſeries.

SCHOLIUM.

The formulæ in this theorem may likewiſe be expreſſed in the following manner, with the negative and poſitive ſign alternately after the firſt term— Thus:

$$\left.\begin{array}{l}
\text{1ſt, } A, - A.\dfrac{D-C}{B} - A.\dfrac{F-E}{D} - \text{\&c. to } \left(\dfrac{p-1}{1}\right)\text{ terms, } + A.\dfrac{D-C}{D}.\dfrac{F-E}{F} + \text{\&c. to } \left(\dfrac{p-1}{1}.\dfrac{p-2}{2}\right)\text{ terms, } - + \text{\&c. \&c.}\\
\text{2d, or } B, - B.\dfrac{B-A}{B} - B.\dfrac{D-C}{D} + \text{\&c. to } \left(\dfrac{p}{1}\right)\text{ terms, } + B.\dfrac{B-A}{B}.\dfrac{D-C}{D} + \text{\&c. to } \left(\dfrac{p}{1}.\dfrac{p-1}{2}\right)\text{ terms, } - + \text{\&c. \&c.}
\end{array}\right\}$$

has to B, the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $\frac{p-1}{1}$, with the ratio of A to B.

And

$$\left.\begin{array}{l}
\text{3d, } B, - B.\dfrac{D-C}{D} - B.\dfrac{F-E}{F} - \text{\&c. to } \left(\dfrac{p-1}{1}\right)\text{ terms, } + B.\dfrac{D-C}{D}.\dfrac{F-E}{F} + \text{\&c. to } \left(\dfrac{p-1}{1}.\dfrac{p-2}{2}\right)\text{ terms, } - + \text{\&c. \&c.}\\
\text{4th, or } A, - A.\dfrac{A-B}{A} - A.\dfrac{D-C}{D} - \text{\&c. to } \left(\dfrac{p}{1}\right)\text{ terms, } + A.\dfrac{A-B}{A}.\dfrac{D-C}{D} + \text{\&c. to } \left(\dfrac{p}{1}.\dfrac{p-1}{2}\right)\text{ terms, } - + \text{\&c. \&c.}
\end{array}\right\}$$

has to A, the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $\left(\frac{p-1}{1}\right)$, with the ratio of B to A.

The ſame mode of expreſſion, which, on many occaſions, will be found uſeful, holds likewiſe with regard to the formulæ in the Corollaries.

If for A and B, in the firſt two formulæ of the theorem, we ſubſtitute $\frac{A^{p-1}}{A^{p-2}}$ and $\frac{B^{p}}{B^{p-1}}$, which are reſpectively equal to them, and for A — B, C — D, E — F, B — A, D — C, F — E, &c. put Γ, Δ, Θ, Λ, Π, Φ, &c. we ſhall evidently have them expreſſed thus:

$$\left.\begin{array}{l}
\text{5th, } \dfrac{A^{p-1}, + A^{p-1}.\dfrac{\Delta}{D} + A^{p-1}.\dfrac{\Theta}{F} + \text{\&c. to } \left(\dfrac{p-1}{1}\right)\text{ terms, } + A^{p-1}.\dfrac{\Delta}{D}.\dfrac{\Theta}{F} + \text{\&c. to } \left(\dfrac{p-1}{1}.\dfrac{p-2}{2}\right)\text{ terms, } + \text{\&c. \&c.}}{A^{p-2}}\\
\text{6th, } \dfrac{\text{or } A^{p-1}, - A^{p-1}.\dfrac{\Pi}{D} - A^{p-1}.\dfrac{\Phi}{F} - \text{\&c to } \left(\dfrac{p-1}{1}\right)\text{ terms, } + A^{p-1}.\dfrac{\Pi}{D}.\dfrac{\Phi}{F} + \text{\&c. to } \left(\dfrac{p-1}{1}.\dfrac{p-2}{2}\right)\text{ terms, } - + \text{\&c. \&c.}}{A^{p-2}}\\
\text{7th, } \dfrac{B^{p} + B^{p}.\dfrac{\Gamma}{B} + B^{p}.\dfrac{\Delta}{D} + B^{p}.\dfrac{\Theta}{F} + \text{\&c. to } \left(\dfrac{p}{1}\right)\text{ terms, } + B^{p}.\dfrac{\Gamma}{B}.\dfrac{\Delta}{D} + \text{\&c. to } \left(\dfrac{p}{1}.\dfrac{p-1}{2}\right)\text{ terms, } + \text{\&c. \&c.}}{B^{p-1}}\\
\text{8th, } \dfrac{\text{or } B^{p}, - B^{p}.\dfrac{\Lambda}{B} - B^{p}.\dfrac{\Pi}{D} - B^{p}.\dfrac{\Phi}{F} - \text{\&c. to } \left(\dfrac{p}{1}\right)\text{ terms, } + B^{p}.\dfrac{\Lambda}{B}.\dfrac{\Pi}{D} + \text{\&c. to } \left(\dfrac{p}{1}.\dfrac{p-1}{2}\right)\text{ terms, } - + \text{\&c. \&c.}}{B^{p-1}}
\end{array}\right\}$$

having each of them to B the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of A to B.

Alſo, if for B and A, in the third and fourth formulæ of the theorem, we ſubſtitute $\frac{B^{p-1}}{B^{p-2}}$ and $\frac{A^{p}}{A^{p-1}}$, which are reſpectively equal to them, we have them expreſſed thus:

9th, $$\frac{B^{p-1},+B^{p-1}.\frac{\Delta}{D}+B^{p-1}.\frac{\Theta}{F}+\&c.\text{ to }(\frac{p-1}{1})\text{ terms},+B^{p-1}.\frac{\Delta}{D}.\frac{\Theta}{F}+\&c.\text{ to }(\frac{p-1}{1}.\frac{p-2}{2})\text{ terms},+\&c.\ \&c.}{B^{p-2}}$$

10th, $$\text{or }\frac{B^{p-1},-B^{p-1}.\frac{\Pi}{D}-B^{p-1}.\frac{\Phi}{F}-\&c.\text{ to }(\frac{p-1}{1})\text{ terms},+B^{p-1}.\frac{\Pi}{D}.\frac{\Phi}{F}+\&c.\text{ to }(\frac{p-1}{1}.\frac{p-2}{2})\text{ terms},-+\&c.\&c.}{B^{p-2}}$$

11th, $$\frac{A^{p},+A^{p}.\frac{\Lambda}{A}+A^{p}.\frac{\Delta}{D}+A^{p}.\frac{\Theta}{F}+\&c.\text{ to }(\frac{p}{1})\text{ terms},+A^{p}.\frac{\Lambda}{A}.\frac{\Delta}{D}+\&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms},+\&c.\ \&c.}{A^{p-1}}$$

12th, $$\text{or }\frac{A^{p},-A^{p}.\frac{\Gamma}{A}-A^{p}.\frac{\Pi}{D}-A^{p}.\frac{\Phi}{F}-\&c.\text{ to }(\frac{p}{1})\text{ terms},+A^{p}.\frac{\Gamma}{A}.\frac{\Pi}{D}+\&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms},-+\&c.\ \&c.}{A^{p-1}}$$

each of which has to A the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of B to A.

In the ſame manner may the formulæ in the Corollaries be expreſſed—Thus, for inſtance, if, in Cor. 3, for A — M, B — N, M — A, N — B, we ſubſtitute Σ, Υ, Ψ, Ω, reſpectively, we ſhall have the formulæ in it expreſſed thus:

13th, $$\frac{M^{p},+M^{p}.\frac{\Sigma}{M}+M^{p}.\frac{\Delta}{D}+M^{p}.\frac{\Theta}{F}+\&c.\text{ to }(\frac{p}{1}\ \text{erms},+M^{p}.\frac{\Sigma}{M}.\frac{\Delta}{D}+\&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms},+\&c.\ \&c.}{M^{p-1}}$$

14th, $$\text{or }\frac{M^{p},-M^{p}.\frac{\Psi}{M}-M^{p}.\frac{\Pi}{D}-M^{p}.\frac{\Phi}{F}-\&c.\text{ to }(\frac{p}{1})\text{ terms},+M^{p}.\frac{\Psi}{M}.\frac{\Pi}{D}+\&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms},-+\&c.\ \&c.}{M^{p-1}}$$

each of which has to B the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of A to B. And

15th, $$\frac{N^{p},+N^{p}.\frac{\Upsilon}{N}+N^{p}.\frac{\Delta}{D}+N^{p}.\frac{\Theta}{F}+\&c.\text{ to }(\frac{p}{1})\text{ terms},+N^{p}.\frac{\Upsilon}{N}.\frac{\Delta}{D}+\&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms},+\&c.\ \&c.}{N^{p-1}}$$

16th, $$\text{or }\frac{N^{p},-N^{p}.\frac{\Omega}{N}-N^{p}.\frac{\Pi}{D}-N^{p}.\frac{\Phi}{F}-\&c.\text{ to }(\frac{p}{1})\text{ terms},+N^{p}.\frac{\Omega}{N}.\frac{\Pi}{D}+\&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms},-+\&c.\ \&c.}{N^{p-1}}$$

each of which has to A the ratio, which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of B to A.

The 7th, 8th, 11th, 12th, 13th, 14th, 15th, and 16th formulæ, in this ſcholium, are geometrical theorems, each of them infinitely more general than another geometrical theorem, which is derivable from any one of them, and of which (when it is ſuppoſed to become numerical, or to expreſs the algebraic values of magnitudes) the famous binomial theorem, given by Sir Iſaac Newton, is only a particular caſe—Thus, for example, in formula 7th in that particular caſe, when Γ, Δ, Θ, &c. are equal among themſelves, B, D, F, &c. likewiſe equal, the expreſſion becomes

$$\frac{B^{p} + \frac{p}{1} B^{p-1} \Gamma + \frac{p}{1} . \frac{p-1}{2} . B^{p-2} \Gamma^{2} + \frac{p}{1} . \frac{p-1}{2} . \frac{p-2}{3} . B^{p-3} \Gamma^{3} + \text{\&c. \&c. to } \Gamma^{p}}{B^{p-1}},$$

having to B, the ratio which is produced by compounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, when each of them is equal to the ratio of A to B, that is, ſuch a multiplicate ratio of A to B, as is expreſſed by the number (p). Wherefore the expreſſion may be denoted by, and is indeed the ſame with, $\frac{A^{p}}{B^{p-1}}$, which ariſes from taking a third proportional to B and A, a fourth proportional to B, A, and this third proportional, and ſo on, till the operation be repeated $(p-1)$ times. But as the magnitude ariſing from this operation is the ſame, whether A be conſidered as a ſingle magnitude of the ſame kind with B, or as compoſed of the aggregate or difference of ſeveral homogeneous magnitudes of the ſame kind with B, if, inſtead of $B + \Gamma = A$, we ſubſtitute the aggregate or difference of any two magnitudes of the ſame kind with A, ſuch as, for inſtance, $V \pm W = A$, (for the ſigns of the even, or 2d, 4th, 6th, &c. terms in the expreſſion are poſitive or negative, according as Γ is additive or ſubtractive, that is, according as A is greater or leſs than B; and $\frac{p}{1}$, $\frac{p}{1} . \frac{p-1}{2}$, &c. depend not on the quantities of the magnitudes, of which the aggregate or difference is equal to A, but on the number of the ratios compounded together) we get the expreſſion

$$\frac{V^{p} \pm \frac{p}{1} . V^{p-1} W + \frac{p}{1} . \frac{p-1}{2} . V^{p-2} W^{2} \pm \frac{p}{1} . \frac{p-1}{2} . \frac{p-2}{3} . V^{p-3} W^{3} + \pm \text{\&c. to } W^{p}}{B^{p-1}},$$

which is a magnitude of the ſame kind with A and B, having to B ſuch a multiplicate ratio of A to B, as is expreſſed by the number (p), and is ſtrictly and literally a geometrical expreſſion, having to B ſuch a multiplicate ratio of V + W to B, or V — W to B (according as V + W is equal to A, or V — W is equal to A), as is denoted by the number (p). Now it is evident that it muſt vary indefinitely with the endleſs variations in the quantity of the magnitude B, its geometrical ſtandard of compariſon, even when A or V ± W is conſidered as invariable—When it is ſuppoſed to become an arithmetical expreſſion, and B to repreſent number, it is likewiſe evident, that it muſt vary indefinitely with the endleſs variations in the quantity of B, its then arithmetical ſtandard of compariſon, even when A or V ± W continues invariable. The magnitude of the expreſſion then, when A is given or conſtant, depends entirely on the quantity or degree of magnitude in the ſtandard, to which it is referred, whether the ſame be geometrical or arithmetical, each ſtandard having a claſs of innumerable different values or magnitudes belonging to itſelf.—There is one claſs, and one alone, in which the expreſſion is not affected by the ſtandard, namely, when B is equal to 1, or arithmetical unity: for then it evidently becomes $\mathrm{v}^{p} \pm \frac{p}{1} . \mathrm{v}^{p-1} . \mathrm{w} + \frac{p}{1} . \frac{p-1}{2} . \mathrm{v}^{p-2} . \mathrm{w}^{2} \pm \frac{p}{1} . \frac{p-1}{2} . \frac{p-2}{3} . \mathrm{v}^{p-3} . \mathrm{w}^{3} + \pm$ &c. to w^{p}, an expreſſion in a ſimilar form to that of the binomial theorem, and having to 1 ſuch a ratio of V ± W to 1, as is denoted by (p). Whence it is manifeſt, that to this claſs, and this alone, the binomial theorem, as delivered by Sir Iſaac, extends, and that any particular caſe of what is commonly called involution, or evolution, is only ſuch a number as has to 1 ſuch a ratio of the number involved or evolved, as is denoted by what is uſually ſtyled its index or exponent of the power. For Nature furniſhes no ideas reſpecting the quantities of magnitudes that are not, by means of compariſon, derivable from, and reducible to, the ideas of augmentation and diminution, of increaſe and decreaſe.

A ſimilar inference and expreſſion may be derived from the ſaid 7th, 8th, 11th, 12th, 13th, 14th, 15th, or 16th formula, in this ſcholium, when the ratios of C to D, E to F, G to H, &c. inſtead of being equal to the ratio of A to B, are equal to any multiples, or ſubmultiples, of it, or have to it any ratios whatever. This, however, I will take an opportunity of conſidering and illuſtrating in the ſubſequent parts of the paper.

THEOREM II.

The ſame things being ſuppoſed as in theorem 1, we ſhall have the following formulæ:

1ſt, $A, - A.\frac{C-D}{C} - A.\frac{E-F}{E} - \&c.\text{ to }(\frac{p-1}{1})\text{ terms}, + A.\frac{C-D}{C}.\frac{E-F}{E} + \&c.\text{ to }(\frac{p-1}{1}.\frac{p-2}{2})\text{ terms}, - + \&c.\ \&c.$

2d, or $A, + A.\frac{D-C}{C} + A.\frac{F-E}{E} + \&c.\text{ to }(\frac{p-1}{1})\text{ terms}, + A.\frac{D-C}{C}.\frac{F-E}{E} + \&c.\text{ to }(\frac{p-1}{1}.\frac{p-2}{2})\text{ terms}, + \&c.\ \&c.$

3d, or $B, - B.\frac{B-A}{B} - B.\frac{C-D}{C} - \&c.\text{ to }(\frac{p}{1})\text{ terms}, + B.\frac{B-A}{B}.\frac{C-D}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, - + \&c.\ \&c.$

4th, or $B, + B.\frac{A-B}{B} + B.\frac{D-C}{C} + \&c.\text{ to }(\frac{p}{1})\text{ terms}, + B.\frac{A-B}{B}.\frac{D-C}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, + \&c.\ \&c.$

5th, or $M, - M.\frac{M-A}{M} - M.\frac{C-D}{C} - \&c.\text{ to }(\frac{p}{1})\text{ terms}, + M.\frac{M-A}{M}.\frac{C-D}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, - + \&c.\ \&c.$

6th, or $M, + M.\frac{A-M}{M} + M.\frac{D-C}{C} + \&c.\text{ to }(\frac{p}{1})\text{ terms}, + M.\frac{A-M}{M}.\frac{D-C}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, + \&c.\ \&c.$

7th, $$\frac{\text{or } A^{p-1}, - A^{p-1}.\frac{\Delta}{C} - A^{p-1}.\frac{\Theta}{E} - \&c.\text{ to }(\frac{p-1}{1})\text{ terms}, + A^{p-1}.\frac{\Delta}{C}.\frac{\Theta}{E} + \&c.\text{ to }(\frac{p-1}{1}.\frac{p-2}{2})\text{ terms}, - + \&c.\ \&c.}{A^{p-2}}$$

8th, $$\frac{\text{or } A^{p-1}, + A^{p-1}.\frac{\Pi}{C} + A^{p-1}.\frac{\Phi}{E} + \&c.\text{ to }(\frac{p-1}{1})\text{ terms}, + A^{p-1}.\frac{\Pi}{C}.\frac{\Phi}{E} + \&c.\text{ to }(\frac{p-1}{1}.\frac{p-2}{2})\text{ terms}, + \&c.\ \&c.}{A^{p-2}}$$

9th, $$\frac{B^{p}, - B^{p}.\frac{\Lambda}{B} - B^{p}.\frac{\Delta}{C} - \&c.\text{ to }(\frac{p}{1})\text{ terms}, + B^{p}.\frac{\Lambda}{B}.\frac{\Delta}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, - + \&c.\ \&c.}{B^{p-1}}$$

10th, $$\frac{\text{or } B^{p}, + B^{p}.\frac{\Gamma}{B} + B^{p}.\frac{\Pi}{C} + \&c.\text{ to }(\frac{p}{1})\text{ terms}, + B^{p}.\frac{\Gamma}{B}.\frac{\Pi}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, + \&c.\ \&c.}{B^{p-1}}$$

11th, $$\frac{\text{or } M^{p}, - M^{p}.\frac{\Psi}{M} - M^{p}.\frac{\Delta}{C} - \&c.\text{ to }(\frac{p}{1})\text{ terms}, + M^{p}.\frac{\Psi}{M}.\frac{\Delta}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, - + \&c.\&c.}{M^{p-1}}$$

12th, $$\frac{\text{or } M^{p}, + M^{p}.\frac{\Sigma}{M} + M^{p}.\frac{\Pi}{C} + \&c.\text{ to }(\frac{p}{1})\text{ terms}, + M^{p}.\frac{\Sigma}{M}.\frac{\Pi}{C} + \&c.\text{ to }(\frac{p}{1}.\frac{p-1}{2})\text{ terms}, + \&c.\ \&c.}{M^{p-1}}$$

having each of them to B the ratio, which is produced by decompounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of A to B; or by compounding their inverſe, viz. the ratios of D to C, F to E, H to G, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of A to B. M is ſuppoſed to be any magnitude whatever of the ſame kind with A and B, whether it be greater or leſs than A. And

13th, $B, - B.\frac{C-D}{C} - B.\frac{E-F}{E} -$ &c. to $(\frac{p-1}{1})$ terms, $+ B.\frac{C-D}{C}.\frac{E-F}{E} +$ &c. to $(\frac{p-1}{1}.\frac{p-2}{2})$ terms, $- +$ &c. &c.

14th, or $B, + B.\frac{D-C}{C} + B.\frac{F-E}{E} +$ &c. to $(\frac{p-1}{1})$ terms, $+ B.\frac{D-C}{C}.\frac{F-E}{E} +$ &c. to $(\frac{p-1}{1}.\frac{p-2}{2})$ terms, $+$ &c. &c.

15th, or $A, - A.\frac{A-B}{A} - A.\frac{C-D}{C} -$ &c. to $(\frac{p}{1})$ terms, $+ A.\frac{A-B}{A}.\frac{C-D}{C} +$ &c. to $(\frac{p}{1}.\frac{p-1}{2})$ terms, $- +$ &c. &c.

16th, or $A, + A.\frac{B-A}{A} + A.\frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ A.\frac{B-A}{A}.\frac{D-C}{C} +$ &c. to $(\frac{p}{1}.\frac{p-1}{2})$ terms, $+$ &c. &c.

17th, or $N, - N.\frac{N-B}{N} - N.\frac{C-D}{C} -$ &c. to $(\frac{p}{1})$ terms, $+ N.\frac{N-B}{N}.\frac{C-D}{C} +$ &c. to $(\frac{p}{1}.\frac{p-1}{2})$ terms, $- +$ &c. &c.

18th, or $N, + N.\frac{B-N}{N} + N.\frac{D-C}{C} +$ &c. to $(\frac{p}{1})$ terms, $+ N.\frac{B-N}{N}.\frac{D-C}{C} +$ &c. to $(\frac{p}{1}.\frac{p-1}{2})$ terms, $+$ &c. &c.

19th, $$\frac{\text{or } B^{p-1}, - B^{p-1}.\frac{\Delta}{C} - B^{p-1}.\frac{\Theta}{E} - \text{\&c. to } (\frac{p-1}{1}) \text{ terms}, + B^{p-1}.\frac{\Delta}{C}.\frac{\Theta}{E} + \text{\&c. to } (\frac{p-1}{1}.\frac{p-2}{2}) \text{ terms}, - + \text{\&c. \&c.}}{B^{p-2}}$$

20th, $$\frac{\text{or } B^{p-1}, + B^{p-1}.\frac{\Pi}{C} + B^{p-1}.\frac{\Phi}{E} + \text{\&c. to } (\frac{p-1}{1}) \text{ terms}, + B^{p-1}.\frac{\Pi}{C}.\frac{\Phi}{E} + \text{\&c. to } (\frac{p-1}{1}.\frac{p-2}{2}) \text{ terms}, + \text{\&c. \&c.}}{B^{p-2}}$$

21ſt, $$\frac{\text{or } A^{p}, - A^{p}.\frac{\Gamma}{A} - A^{p}.\frac{\Delta}{C} - \text{\&c. to } (\frac{p}{1}) \text{ terms}, + A^{p}.\frac{\Gamma}{A}.\frac{\Delta}{C} + \text{\&c. to } (\frac{p}{1}.\frac{p-1}{2}) \text{ terms}, - + \text{\&c. \&c.}}{A^{p-1}}$$

22d, $$\frac{\text{or } A^{p}, + A^{p}.\frac{\Lambda}{A} + A^{p}.\frac{\Pi}{C} + \text{\&c. to } (\frac{p}{1}) \text{ terms}, + A^{p}.\frac{\Lambda}{A}.\frac{\Pi}{C} + \text{\&c. to } (\frac{p}{1}.\frac{p-1}{2}) \text{ terms}, + \text{\&c. \&c.}}{A^{p-1}}$$

23d, $$\frac{\text{or } N^{p}, - N^{p}.\frac{\Omega}{N} - N^{p}.\frac{\Delta}{C} - \text{\&c. to } (\frac{p}{1}) \text{ terms}, + N^{p}.\frac{\Omega}{N}.\frac{\Delta}{C} + \text{\&c. to } (\frac{p}{1}.\frac{p-1}{2}) \text{ terms}, - + \text{\&c. \&c.}}{N^{p-1}}$$

24th, $$\frac{\text{or } N^{p}, + N^{p}.\frac{\Upsilon}{N} + N^{p}.\frac{\Pi}{C} + \text{\&c. to } (\frac{p}{1}) \text{ terms}, + N^{p}.\frac{\Upsilon}{N}.\frac{\Pi}{C} + \text{\&c. to } (\frac{p}{1}.\frac{p-1}{2}) \text{ terms}, + \text{\&c. \&c.}}{N^{p-1}}$$

has to A the ratio, which is produced by decompounding the ratios of C to D, E to F, G to H, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of B to A, or by compounding their inverſe, viz. the ratios of D to C, F to E, H to G, &c. continued to the number $(\frac{p-1}{1})$, with the ratio of B to A.

SCHOLIUM.

I have ſet down the preceding variety of formulæ, in this theorem, not that I thought it abſolutely neceſſary to do ſo for thoſe perſons who underſtand thoroughly the geometrical principles from which they are derived, but becauſe the ſame expreſſion is frequently found more uſeful & commodious in one form than in another. Many inferences or corollaries may be drawn from them, of the greateſt uſe, and moſt extenſive application. To enable one to draw ſuch, and apply them readily, it may not perhaps be unneceſſary to obſerve; that, if in the expreſſions, (in theorem 1.) which have to B the ratio, that is produced by compounding the ratios of C to D, E to F, G to H, &c, continued to the number ($\frac{p-1}{1}$), with the ratio of A to B, for A, C, E, G, &c, there be ſubſtituted B, D, F, H, &c, and for B, D, F, H, &c, there be ſubſtituted A, C, E, G, &c, there ariſe expreſſions to which A has the ſame ratio reſpectively; and if in thoſe expreſſions, which have to A the ratio that is produced by compounding the ſaid ratios of C to D, E to F, G to H, &c, continued to the number ($\frac{p-1}{1}$), with the ratio of B to A, for B, C, E, G, &c, there be ſubſtituted A, D, F, H, &c, and for A, D, F, H, &c, there be ſubſtituted B, C, E, G, &c, there will ariſe expreſſions, to which B has the ſame ratio reſpectively; alſo that, if in the expreſſions (in this theorem, viz. theorem 2), which have to B the ratio, that is produced by decompounding the ratios of C to D, E to F, G to H, &c, continued to the number ($\frac{p-1}{1}$), or by compounding their inverſe, viz. the ratios of D to C, F to E, H to G, &c, with the ratio of A to B, for A, D, F, H, &c, there be ſubſtituted B, C, E, G, &c, and for B, C, E, G, &c, there be ſubſtituted A, D, F, H, &c, there will ariſe expreſſions, to which A has the ſame ratio reſpectively; and if in thoſe expreſſions, which have to A the ratio, that is produced by decompounding the ratios of C to D, E to F, G to H, &c, continued to the number ($\frac{p-1}{1}$), or by compounding their inverſe, viz. the ratios of D to C, F to E, H to G, &c, continued to the ſame number ($\frac{p-1}{1}$), with the ratio of B to A, for B, D, F, H, &c, there be ſubſtituted A, C, E, G, &c, and for A, C, E, G, &c, there be ſubſtituted B, D, F, H, &c, there will ariſe expreſſions to which B has the ſame ratio reſpectively. If this circumſtance be ſufficiently attended to, numberleſs inferences or corollaries may be drawn from them, of the greateſt poſſible uſe and advantage in many inveſtigations, and particularly in the management of ſeries. It is likewiſe evident, that, although the expreſſions always vary with the ſtandards of compariſon, in ſuch circumſtances, or when reduced to ſuch particular forms, as I have taken notice of in the ſcholium to theorem 1, namely, either when the conſequent of every ratio compounded, is ſuppoſed to be always in a ſtate of equality with the ſtandard, or when the ratio of any magnitude to the ſtandard, is compounded any number of times with itſelf, or with any number of ratios reſpectively equal to or the ſame with itſelf, yet the magnitudes of the expreſſions in all other circumſtances, and in their general forms, are not regulated or affected in the ſmalleſt degree by the ſtandard of compariſon, or the conſequent of the firſt ratio, or of that with which the other ratios are compounded—And the reaſon is plain—In any expreſſion the conſequent of the firſt ratio, with which the others are compounded, does not enter into the formation of it at all, ſtrictly ſpeaking, or at leaſt, when it does, only by way of ſubſtitution without reference or relation to any ſtandard whatever. Thus, for inſtance, (B) the conſequent of the firſt ratio, viz, of A to B, does not enter into the formation of the 1ſt formula in theorem 1, or of the 1ſt and 2d formulæ of this theorem, which, without ſuffering the leaſt variation in their quantities, may evidently be referred to any other magnitude (M) whatever, as a ſtandard, as well as to (B), that enters into the formation of the 2d formula in theorem 1, and of the 3d and 4th formulæ in this theorem, by way of ſubſtitution only, when for (A) there is put its equal $B + B . \frac{A - B}{B}$.

THEOREM III.

Let the ratios of C to D, E to F, G to H, &c, have reſpectively to the ratio of A to B, the ratios of I to K, L to M, N to O, &c; and let P and Q, S and T, taken two and two, be homogeneous magnitudes, of any kind; alſo let P together with a fourth proportional to K, I and P, together with a fourth proportional to M, L and P, together with a fourth proportional to O, N and P, &c, or $P + P . \frac{I}{K} + P . \frac{L}{M} + P . \frac{N}{O} +$ &c. be repreſented by R. Then

1ſt. $A+\frac{R-Q}{Q}\cdot A\cdot\frac{A-B}{B}+\frac{R-Q}{Q}\cdot\frac{R-2Q}{2Q}\cdot A\cdot\left.\overline{\frac{A-B}{B}}\right|^{2}+\frac{R-Q}{Q}\cdot\frac{R-2Q}{2Q}\cdot\frac{R-3Q}{3Q}\cdot A\cdot\left.\overline{\frac{A-B}{B}}\right|^{3}+\text{\&c. \&c.}$

2d, or $A-\frac{R-Q}{Q}\cdot A\cdot\frac{B-A}{B}+\frac{R-Q}{Q}\cdot\frac{R-2Q}{2Q}\cdot A\cdot\left.\overline{\frac{B-A}{B}}\right|^{2}-\frac{R-Q}{Q}\cdot\frac{R-2Q}{2Q}\cdot\frac{R-3Q}{3Q}\cdot A\cdot\left.\overline{\frac{B-A}{B}}\right|^{3}+-\text{\&c. \&c.}$

3d, or $A+\frac{R-Q}{Q}\cdot W\cdot\frac{A-B}{B}+\frac{R-2Q}{2Q}\cdot X\cdot\frac{A-B}{B}+\frac{R-3Q}{3Q}\cdot Y\cdot\frac{A-B}{B}+\text{\&c. \&c.}$

4th, or $A-\frac{R-Q}{Q}\cdot W\cdot\frac{B-A}{B}+\frac{R-2Q}{2Q}\cdot X\cdot\frac{B-A}{B}-\frac{R-3Q}{3Q}\cdot Y\cdot\frac{B-A}{B}+-\text{\&c. \&c.}$

ſuppoſing W, X, Y, &c, to repreſent the 1ſt, 2d, 3d, &c. terms;

5th, or $B+\frac{R}{Q}\cdot B\cdot\frac{A-B}{B}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B\cdot\left.\overline{\frac{A-B}{B}}\right|^{2}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B\cdot\left.\overline{\frac{A-B}{B}}\right|^{3}+\text{\&c. \&c.}$

6th, or $B-\frac{R}{Q}\cdot B\cdot\frac{B-A}{B}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B\cdot\left.\overline{\frac{B-A}{B}}\right|^{2}-\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B\cdot\left.\overline{\frac{B-A}{B}}\right|^{3}+-\text{\&c. \&c.}$

7th, or $B+\frac{R}{Q}\cdot I\cdot\frac{A-B}{B}+\frac{R-Q}{2Q}\cdot K\cdot\frac{A-B}{B}+\frac{R-2Q}{3Q}\cdot L\cdot\frac{A-B}{B}+\text{\&c. \&c.}$

8th, or $B-\frac{R}{Q}\cdot I\cdot\frac{B-A}{B}+\frac{R-Q}{2Q}\cdot K\cdot\frac{B-A}{B}-\frac{R-2Q}{3Q}\cdot L\cdot\frac{B-A}{B}+-\text{\&c. \&c.}$

ſuppoſing I, K, L, &c, to repreſent the 1ſt, 2d, 3d, &c, terms;

9th, or $\dfrac{B^{\frac{R}{Q}}+\frac{R}{Q}\cdot B^{\frac{R}{Q}}\cdot\frac{A-B}{B}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B^{\frac{R}{Q}}\cdot\left.\overline{\frac{A-B}{B}}\right|^{2}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B^{\frac{R}{Q}}\cdot\left.\overline{\frac{A-B}{B}}\right|^{3}+\text{\&c. \&c.}}{B^{\frac{R-Q}{Q}}}$

10th, or $\dfrac{B^{\frac{R}{Q}}-\frac{R}{Q}\cdot B^{\frac{R}{Q}}\cdot\frac{B-A}{B}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B^{\frac{R}{Q}}\cdot\left.\overline{\frac{B-A}{B}}\right|^{2}-\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B^{\frac{R}{Q}}\cdot\left.\overline{\frac{A-B}{B}}\right|^{3}+-\text{\&c. \&c.}}{B^{\frac{R-Q}{Q}}}$

11th, or $\dfrac{B^{\frac{R}{Q}}+\frac{R}{Q}\cdot B^{\frac{R-Q}{Q}}\cdot\Gamma+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B^{\frac{R-2Q}{Q}}\cdot\Gamma^{2}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B^{\frac{R-3Q}{Q}}\cdot\Gamma^{3}+\text{\&c. \&c.}}{B^{\frac{R-Q}{Q}}}$

12th, or $\dfrac{B^{\frac{R}{Q}}-\frac{R}{Q}\cdot B^{\frac{R-Q}{Q}}\cdot\Lambda+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B^{\frac{R-2Q}{Q}}\cdot\Lambda^{2}-\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B^{\frac{R-3Q}{Q}}\cdot\Lambda^{3}+-\text{\&c. \&c.}}{B^{\frac{R-Q}{Q}}}$

13th, or $\dfrac{B^{\frac{R}{Q}}+\frac{R}{Q}\cdot B^{\frac{R}{Q}}\cdot\chi+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B^{\frac{R}{Q}}\cdot\chi^{2}+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B^{\frac{R}{Q}}\cdot\chi^{3}+\text{\&c. \&c.}}{B^{\frac{R-Q}{Q}}}$

14th, or $\dfrac{B^{\frac{R}{Q}}-\frac{R}{Q}\cdot B^{\frac{R}{Q}}\cdot\zeta+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot B^{\frac{R}{Q}}\cdot\zeta^{2}-\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot B^{\frac{R}{Q}}\cdot\zeta^{3}+-\text{\&c. \&c.}}{B^{\frac{R-Q}{Q}}}$

ſuppoſing χ, ζ, to be equal to $\frac{\Gamma}{B}$, $\frac{\Lambda}{B}$, reſpectively;

$$\text{15th, or } \frac{V^{\frac{R}{Q}} + \frac{R}{Q}.V^{\frac{R-Q}{Q}}.Z + \frac{R}{Q}.\frac{R-Q}{2Q}.V^{\frac{R-2Q}{Q}}.Z^{2} + \frac{R}{Q}.\frac{R-Q}{2Q}.\frac{R-2Q}{3Q}.V^{\frac{R-3Q}{Q}}.Z^{3} + \&c.\ \&c.}{B^{\frac{R-Q}{Q}}}$$

ſuppoſing $V + Z = A$;

$$\text{16th, or } \frac{M^{\frac{R}{Q}} - \frac{R}{Q}.M^{\frac{R-Q}{Q}}.N + \frac{R}{Q}.\frac{R-Q}{2Q}.M^{\frac{R-2Q}{Q}}.N^{2} - \frac{R}{Q}.\frac{R-Q}{2Q}.\frac{R-2Q}{3Q}.M^{\frac{R-3Q}{Q}}.N^{3} + - \&c.\ \&c.}{B^{\frac{R-Q}{Q}}}$$

ſuppoſing $M - N = A$;

$$\text{17th, or } \overline{\tfrac{S}{T}}\Big|^{\frac{R}{Q}} \times \underline{B + \frac{R}{Q}.B.\frac{\frac{T}{S}.A-B}{B} + \frac{R}{Q}.\frac{R-Q}{2Q}.B.\overline{\frac{\frac{T}{S}.A-B}{B}}\Big|^{2} + \&c.\ \&c.}$$

$$\text{18th, or } \overline{\tfrac{S}{T}}\Big|^{\frac{R}{Q}} \times \frac{T}{S}.A + \frac{R-Q}{Q}.\frac{T}{S}.A.\frac{\frac{T}{S}.A-B}{B} + \frac{R-Q}{Q}.\frac{R-2Q}{2Q}.\frac{T}{S}.A.\overline{\frac{\frac{T}{S}.A-B}{B}}\Big|^{2} + \&c.\ \&c.$$

or &c.

or &c.

has to B the ratio, which has to the ratio that is produced by compounding the ratios of C to D, E to F, G to H, &c, with the ratio of A to B, the ratio of P to Q, or has to B a ratio, which has to the ratio of A to B, the ratio of R to Q, or has to any other magnitude whatever of the ſame kind with A and B, a ratio equal to the ratio (which has to the ratio that is produced by compounding the ratios of C to D, E to F, G to H, &c, with the ratio of A to the magnitude B, the ratio of P to Q), compounded with the ratio of B to that magnitude, or a ratio equal to the ratio (which has to the ratio of A to B the ratio of R to Q), compounded with the ratio of B to that magnitude.

SCHOLIUM.

From the equality of theſe various formulæ, many very uſeful propoſitions and corollaries may at any time be deduced. If for A and B, in them, there be ſubſtituted B and A reſpectively, they will become expreſſions or formulæ, to which A has the ſame ratio, that they have to B, a circumſtance, which will likewiſe furniſh occaſionally numberleſs inferences of the greateſt convenience and uſe. The ratio, which any one of theſe formulæ has to B, compounded with the ratio of B to any other magnitude M of the ſame kind, is equal to the ratio, which the formulæ themſelves have reſpectively to M. If for R in them there be put R + Q, they will be changed into formulæ having the ſame ratio to A, that they themſelves have to B. And if for R there be ſubſtituted R + Q, and in the place of A, where it ſtands ſingly without any magnitude taken from it by the ſign —, A — B and B — A continuing as they are, there be put any other magnitude M, whether it be of the ſame kind with A and B or not, they will be converted into formulæ having the ſame ratio to M.

From what has been delivered, it evidently appears that $\frac{A^{\frac{m}{n}}}{B^{\frac{m-n}{n}}}$ is an expreſſion, ſtrictly and literally ſpeaking, geometrical, namely, a magnitude having to B a ratio, which has to the ratio of A to B, the ratio of *m* to *n*, or having to any other magnitude M of the ſame kind with A and B, the ratio which is produced by compounding the ratio of B to M, with the ratio that has to the ratio of A to B, the ratio of *m* to *n*, and is truly expreſſed by

$$A + \frac{m-n}{n} . A . \frac{A-B}{B} + \frac{m-n}{n} . \frac{m-2n}{2n} . A . \left.\overline{\frac{A-B}{B}}\right|^2 + \frac{m-n}{n} . \frac{m-2n}{2n} . \frac{m-3n}{3n} . A . \left.\overline{\frac{A-B}{B}}\right|^3 + \text{\&c. \&c.}$$

$$\text{or } B + \frac{m}{n} . B . \frac{A-B}{B} + \frac{m}{n} . \frac{m-n}{2n} . B . \left.\overline{\frac{A-B}{B}}\right|^2 + \frac{m}{n} . \frac{m-n}{2n} . \frac{m-2n}{3n} . B . \left.\overline{\frac{A-B}{B}}\right|^3 + \text{\&c. \&c.}$$

$$\text{or } \frac{P^{\frac{m}{n}} \pm \frac{m}{n} . C . \frac{Q}{B} + \frac{m-n}{2n} . D . \frac{Q}{B} \pm \frac{m-2n}{3n} . E . \frac{Q}{B} + \frac{m-3n}{4n} . F . \frac{Q}{B} \pm \frac{m-4n}{5n} . G . \frac{Q}{B} + \pm \text{\&c.}}{B^{\frac{m-n}{n}}}$$

ſuppoſing $P \pm P . \frac{Q}{B}$ to be equal to the magnitude A, C to repreſent $P^{\frac{m}{n}}$, D to repreſent $\frac{m}{n} . C . \frac{Q}{B}$, E to repreſent $\frac{m-n}{2n} . D . \frac{Q}{B}$, &c, &c, and ſo on; that $\frac{M A^{\frac{m}{n}}}{B^{\frac{m}{n}}}$ is alſo a geometrical magnitude, being a fourth proportional to B, the magnitude $\frac{A^{\frac{m}{n}}}{B^{\frac{m-n}{n}}}$ juſt repreſented, and M, and may be expreſſed by $M + \frac{m}{n} . M . \frac{A-B}{B} + \frac{m}{n} . \frac{m-n}{2n} . M . \left.\overline{\frac{A-B}{B}}\right|^2 + \frac{m}{n} . \frac{m-n}{2n} . \frac{m-2n}{3n} . M . \left.\overline{\frac{A-B}{B}}\right|^3 + \text{\&c. \&c.}$

$$\text{or } M \times \frac{P\frac{m}{n} \pm \frac{m}{n} . C . \frac{Q}{B} + \frac{m-n}{2n} . D . \frac{Q}{B} \pm \frac{m-2n}{3n} . E . \frac{Q}{B} + \frac{m-3n}{4n} . F . \frac{Q}{B} \pm \frac{m-4n}{5n} . G . \frac{Q}{B} + \pm \text{\&c.}}{B^{\frac{m}{n}}}$$

ſuppoſing as before $\text{P} \pm \text{C} \cdot \frac{\text{Q}}{\text{B}} = \text{A}$, C to ſtand for $\text{P}^{\frac{m}{n}}$, D for $\frac{m}{n} \cdot \text{C} \cdot \frac{\text{Q}}{\text{B}}$, &c. &c. or &c. &c.—In like manner, $\frac{\text{M}^2\, \text{A}^{\frac{m}{n}}}{\text{B}^{\frac{m}{n}}}$, $\frac{\text{M}^3 .\, \text{A}^{\frac{m}{n}}}{\text{B}^{\frac{m}{n}}}$, may be geometrical magnitudes, the firſt being a rectangle contained by M, or the line repreſenting it, and $\frac{\text{M}\,\text{A}^{\frac{m}{n}}}{\text{B}^{\frac{m}{n}}}$, or the line repreſenting it, and the ſecond a ſolid contained by M, M and $\frac{\text{M}\,\text{A}^{\frac{m}{n}}}{\text{B}^{\frac{m}{n}}}$, or the lines repreſenting them—Alſo A, A^2, A^3, $\text{A}\,\text{M}^2$ &c, may be geometrical magnitudes—But A^4, A^5, $\text{A}^2\,\text{M}^2$, $\text{A}^{\frac{m}{n}}$ &c, cannot be geometrical expreſſions, but muſt denote arithmetical magnitudes, all of which are homogeneous, or of the ſame kind—For geometry, and even nature, knows no diverſity of dimenſions, except the three that go to the formation of ſolidity—Such an expreſſion then as $a^{\frac{m}{n}}$, is only an arithmetical one, having to 1, or unit, ſuch a ratio, as has to the ratio of a to 1, the ratio of m to n, and is indeed truly expreſſed by

$$a + \frac{m-n}{n} . a . \frac{a-1}{1} + \frac{m-n}{n} . \frac{m-2n}{2n} . a . \left.\overline{\frac{a-1}{1}}\right|^2 + \frac{m-n}{n} . \frac{m-2n}{2n} . \frac{m-3n}{3n} . a . \left.\overline{\frac{a-1}{1}}\right|^3 + \text{\&c. \&c.}$$

$$\text{or } 1 + \frac{m}{n} . 1 . \frac{a-1}{1} + \frac{m}{n} . \frac{m-n}{2n} . 1 . \left.\overline{\frac{a-1}{1}}\right|^2 + \frac{m}{n} . \frac{m-n}{2n} . \frac{m-2n}{3n} . 1 . \left.\overline{\frac{a-1}{1}}\right|^3 + \frac{m}{n} . \frac{m-n}{2n} . \frac{m-2n}{3n} . \frac{m-3n}{4n} . 1 . \left.\overline{\frac{a-1}{1}}\right|^4 + \text{\&c.}$$

$$\text{or } p^{\frac{m}{n}} \pm \frac{m}{n} . cq + \frac{m-n}{2n} . dq \pm \frac{m-2n}{3n} . eq + \frac{m-3n}{4n} . fq \pm \frac{m-4n}{5n} . gq + \frac{m-5n}{6n} . hq \pm \text{\&c.}$$

ſuppoſing $p \pm pq = a$, c to repreſent the firſt term, d the ſecond, and ſo on, or &c. &c. Indeed the geometrical expreſſion $\frac{\text{A}^{\frac{m}{n}}}{\text{B}^{\frac{m-n}{n}}}$, in becoming numerical, when (a) is ſuppoſed to repreſent the arithmetical value of A, and (b) that of B, to denote 1 or unit, gives $a^{\frac{m}{n}}$. In like manner $a^{\frac{\sqrt[4]{5}}{\sqrt{3}}}$ is a number having to 1, ſuch a ratio, as has to the ratio of a to 1, the ratio of $\sqrt[4]{5}$ to $\sqrt{3}$, and may be truly expreſſed by

$$a+\frac{\sqrt[4]{5}-\sqrt{3}}{\sqrt{3}}.a.\frac{a-1}{1}+\frac{\sqrt[4]{5}-\sqrt{3}}{\sqrt{3}}.\frac{\sqrt[4]{5}-2.\sqrt{3}}{2.\sqrt{3}}.a.\left.\overline{\frac{a-1}{1}}\right|^{2}+\frac{\sqrt[4]{5}-\sqrt{3}}{\sqrt{3}}.\frac{\sqrt[4]{5}-2.\sqrt{3}}{2.\sqrt{3}}.\frac{\sqrt[4]{5}-3.\sqrt{3}}{3.\sqrt{3}}.a.\left.\overline{\frac{a-1}{1}}\right|^{3}+\&c.$$

$$\text{or }1+\frac{\sqrt[4]{5}}{\sqrt{3}}.1.\frac{a-1}{1}+\frac{\sqrt[4]{5}}{\sqrt{3}}.\frac{\sqrt[4]{5}-\sqrt{3}}{2.\sqrt{3}}.1.\left.\overline{\frac{a-1}{1}}\right|^{2}+\frac{\sqrt[4]{5}}{\sqrt{3}}.\frac{\sqrt[4]{5}-\sqrt{3}}{2.\sqrt{3}}.\frac{\sqrt[4]{5}-2.\sqrt{3}}{3.\sqrt{3}}.1.\left.\overline{\frac{a-1}{1}}\right|^{3}+\&c.\ \&c.$$

$$\text{or }p^{\frac{\sqrt[4]{5}}{\sqrt{3}}}\pm\frac{\sqrt[4]{5}}{\sqrt{3}}.cq+\frac{\sqrt[4]{5}-\sqrt{3}}{2.\sqrt{3}}.dq\pm\frac{\sqrt[4]{5}-2.\sqrt{3}}{3.\sqrt{3}}.eq+\frac{\sqrt[4]{5}-3.\sqrt{3}}{4.\sqrt{3}}.fq\pm\frac{\sqrt[4]{5}-4.\sqrt{3}}{5.\sqrt{3}}.gq+\pm\&c.$$

ſuppoſing $p \pm pq = a$, and c, d, e, f, g, &c, to repreſent the 1ſt, 2d, 3d, 4th, 5th terms, &c. Alſo $\overline{x+y}^{5}$ is a number having to 1, the quintuplicate ratio of $\overline{x+y}$ to 1, or ſuch a ratio to 1, as has to the ratio of $\overline{x+y}$ to 1, the ratio of 5 to 1; $3x^{5}$ is a multiple by 3 of the number, which has to 1 the quintuplicate ratio of x to 1, or ſuch a ratio to 1, as has to the ratio of x to 1, the ratio of 5 to 1; $6y^{4}$ is a multiple by 6 of the number, which has to 1, the quadruplicate ratio of y to 1, or ſuch a ratio to 1, as has to the ratio of y to 1, the ratio of 4 to 1; and ſo on—Any magnitude then whatever, obtained or expreſſed by involution or evolution in algebra, which is nothing but general arithmetic, is only a number having to 1, ſuch a ratio, as has to the ratio of the root, or the number involved or evolved, to 1, the ratio of what is commonly called its index to 1—But as the index may ſtand to 1 in all the poſſible relations of quantity, the parts of it may be compoſed of any magnitudes whatever, ſuch as lines, ſurfaces, ſolids, as well as numbers; and algebra and arithmetic ought to know ſuch expreſſions as $a^{\sqrt[c]{m}}$, $b^{\frac{\sqrt[c]{m}}{\sqrt[d]{n}}}$, $a^{\frac{P}{Q}}$, &c; $2^{\sqrt[3]{4}}$, $3^{\frac{\sqrt[5]{4}}{\sqrt[4]{3}}}$, $4^{\frac{P}{Q}}$, &c, equally as well as a^{m}, $b^{\frac{m}{n}}$, &c. 2^{4}, $3^{\frac{4}{3}}$, &c. Thus for inſtance $4^{\frac{P}{Q}}$, or the number which has ſuch a ratio to 1, as has to the ratio of 4 to 1, the ratio of P to Q, (P and Q repreſenting either two lines, or two ſurfaces, or two ſolids) is equal to, and is truly repreſented by

$$4+\frac{P-Q}{Q}.4.\frac{4-1}{1}+\frac{P-Q}{Q}.\frac{P-2Q}{2Q}.4.\left.\overline{\frac{4-1}{1}}\right|^{2}+\frac{P-Q}{Q}.\frac{P-2Q}{2Q}.\frac{P-3Q}{3Q}.4.\left.\overline{\frac{4-1}{1}}\right|^{3}+\&c.\ \&c.$$

$$\text{or }1+\frac{P}{Q}.1.\frac{4-1}{1}+\frac{P}{Q}.\frac{P-Q}{2Q}.1.\left.\overline{\frac{4-1}{1}}\right|^{2}+\frac{P}{Q}.\frac{P-Q}{2Q}.\frac{P-2Q}{3Q}.1.\left.\overline{\frac{4-1}{1}}\right|^{3}+\&c.\ \&c.$$

$$\text{or }\frac{P}{3Q}\times(1+\frac{P}{Q}.\frac{1}{3}+\frac{P}{Q}.\frac{P-Q}{2Q}.\frac{1}{3^{2}}+\frac{P}{Q}.\frac{P-Q}{2Q}.\frac{P-2Q}{3Q}.\frac{1}{3^{3}}+\frac{P}{Q}.\frac{P-Q}{2Q}.\frac{P-2Q}{3Q}.\frac{P-3Q}{4Q}.\frac{1}{3^{4}}+\&c.$$

or, &c. &c.

It is likewiſe manifeſt, that, ſince A^{4}, A^{5}, $A^{\frac{m}{n}}$, &c, cannot be geometrical magnitudes, but are arithmetical ones, as I have already obſerved, ſo muſt all ſuch expreſſions as $\frac{A^{\frac{m}{n}}}{B^{\frac{m}{n}}}$ alſo be, when A and B are homogeneous—But $\frac{A^{\frac{m}{n}}}{B^{\frac{m}{n}}}$ is truly expreſſed by

$$1 + \frac{m}{n} \cdot \frac{A-B}{B} + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \left.\overline{\frac{A-B}{B}}\right|^{2} + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} \cdot \left.\overline{\frac{A-B}{B}}\right|^{3} + \text{\&c. \&c.}$$

$$\text{or } 1 + \frac{m}{n} \cdot 1 \cdot Q + \frac{m-n}{2n} \cdot D \cdot Q + \frac{m-2n}{3n} \cdot E \cdot Q + \frac{m-3n}{4n} \cdot F \cdot Q + \frac{m-4n}{5n} \cdot G \cdot Q + \text{\&c.}$$

Q repreſenting $\frac{A-B}{B}$ and I, D, E, F, G, &c, the 1ſt, 2d, 3d, 4th, 5th terms, &c.

$$\text{or } \frac{P^{\frac{m}{n}} \pm \frac{m}{n} \cdot C \cdot \frac{Q}{B} + \frac{m-n}{2n} \cdot D \cdot \frac{Q}{B} \pm \frac{m-2n}{3n} \cdot E \cdot \frac{Q}{B} + \frac{m-3n}{4n} \cdot F \cdot \frac{Q}{B} \pm + \text{\&c. \&c.}}{B^{\frac{m}{n}}}$$

$P \pm P \cdot \frac{Q}{B}$ being ſuppoſed $=$ A, and C, D, E, &c, to repreſent the 1ſt, 2d, 3d terms, &c.

$$\text{or } \frac{A}{B} \times \left(1 + \frac{m-n}{n} \cdot \frac{A-B}{B} + \frac{m-n}{n} \cdot \frac{m-2n}{2n} \cdot \left.\overline{\frac{A-B}{B}}\right|^{2} + \frac{m-n}{n} \cdot \frac{m-2n}{2n} \cdot \frac{m-3n}{3n} \cdot \left.\overline{\frac{A-B}{B}}\right|^{3} + \text{\&c.}\right)$$

or &c. &c.—Now it is evident that, let A and B, P and Q, in the preceding expreſſions, be any magnitudes whatever, $\frac{A-B}{B}$, $\frac{P-Q}{Q}$, &c. can only be algebraical or arithmetical magnitudes.

In like manner, in arithmetic, 2 × 7 is a number having ſuch a ratio to 1, as is equal to the ratio of 7 to 1 compounded with the ratio of 2 to 1, and by means of theorem 1ſt, may be truly expreſſed by $2 + 2 \cdot \frac{7-1}{1}$; and $3 \times 5 \times 8 \times 9$ is a number having to 1 ſuch a ratio, as is equal to the ratios of 5 to 1, 8 to 1, and 9 to 1, compounded with the ratio of 3 to 1, and by the ſame theorem may be truly and accurately expreſſed by $3 + 3 \cdot \frac{5-1}{1} + 3 \cdot \frac{8-1}{1} + 3 \cdot \frac{9-1}{1} + 3 \cdot \frac{5-1}{1} \cdot \frac{8-1}{1} + 3 \cdot \frac{5-1}{1} \cdot \frac{9-1}{1} + 3 \cdot \frac{3-1}{1} \cdot \frac{9-1}{1} + 3 \cdot \frac{5-1}{1} \cdot \frac{8-1}{1} \cdot \frac{9-1}{1}$, or by $3 \times \left(\frac{1+4+7+8}{1} + \frac{4\cdot7+4\cdot8+7\cdot8}{1.1} + \frac{4\cdot7\cdot8}{1.1.1}\right)$ and ſo on; and $\frac{7}{2}$ is a number having ſuch a ratio to 1, as is equal to the ratio of 2 to 1 decompounded, or its inverſe the ratio of 1 to 2 compounded with the ratio of 7 to 1, and by theorem 2d may be expreſſed by $7 - 7 \cdot \frac{2-1}{2}$ or $7 + 7 \cdot \frac{1-2}{2}$; and $\frac{7}{3\cdot5\cdot9}$ is a number having to 1 ſuch a ratio, as is equal to the ratio of 3 to

1, 5 to 1, and 9 to 1, decompounded with, or their inverſe, the ratios of 1 to 3, 1 to 5, and 1 to 9, compounded with the ratio of 7 to 1, and, by means of theorem 2d, may be truly and accurately expreſſed by

$$7 - 7\cdot\frac{3-1}{3} - 7\cdot\frac{5-1}{5} - 7\cdot\frac{9-1}{9} + 7\cdot\frac{3-1}{3}\cdot\frac{5-1}{5} + 7\cdot\frac{3-1}{3}\cdot\frac{9-1}{9} + 7\cdot\frac{5-1}{5}\cdot\frac{9-1}{9} - 7\cdot\frac{3-1}{3}\cdot\frac{5-1}{5}\cdot\frac{9-1}{9},$$

or by $$7 + 7\cdot\frac{1-3}{3} + 7\cdot\frac{1-5}{5} + 7\cdot\frac{1-9}{9} + 7\cdot\frac{1-3}{3}\cdot\frac{1-5}{5} + 7\cdot\frac{1-3}{3}\cdot\frac{1-9}{9} + 7\cdot\frac{1-5}{5}\cdot\frac{1-9}{9} + 7\cdot\frac{1-3}{3}\cdot\frac{1-5}{5}\cdot\frac{1-9}{9},$$

or by $7 \times \left(\frac{1}{1} - \frac{2}{3} - \frac{4}{5} - \frac{8}{9} + \frac{2\cdot 4}{3\cdot 5} + \frac{2\cdot 8}{3\cdot 9} + \frac{4\cdot 8}{5\cdot 9} - \frac{2\cdot 4\cdot 8}{3\cdot 5\cdot 9}\right)$, and ſo on. Alſo, in algebra ab is a number having to 1 ſuch a ratio as is equal to the ratio of a to 1, compounded with the ratio of b to 1, and, by theorem 1ſt, is expreſſed by $a + a\cdot\frac{b-1}{1}$; abc is a number having to 1 ſuch a ratio, as is equal to the ratio, which is compounded of the ratios of a to 1, b to 1, and c to 1, or to the ratio of a to 1, compounded with the ratios of b to 1 and c to 1, and, by theorem 1ſt, is expreſſed by $a + a\cdot\frac{b-1}{1} + a\cdot\frac{c-1}{1} + a\cdot\frac{b-1}{1}\cdot\frac{c-1}{1}$; $abcd$ is a number having to 1 ſuch a ratio, as is equal to the ratio, which is compounded of the ratios of a to 1, b to 1, c to 1, and d to 1, or to the ratio of a to 1 compounded with the ratios of b to 1, c to 1, and d to 1, and, by theorem 1ſt, is expreſſed by $a + a\cdot\frac{b-1}{1} + a\cdot\frac{c-1}{1} + a\cdot\frac{d-1}{1} + a\cdot\frac{b-1}{1}\cdot\frac{c-1}{1} + a\cdot\frac{b-1}{1}\cdot\frac{d-1}{1} + a\cdot\frac{c-1}{1}\cdot\frac{d-1}{1} + a\cdot\frac{b-1}{1}\cdot\frac{c-1}{1}\cdot\frac{d-1}{1}$, or by $a \times \left(\frac{b+c+d-2}{1} + \frac{bc+bd+cd-2b-2c-2d+3}{1.1} + \frac{bcd-bc-bd-cd+b+c+d-1}{1.1.1}\right)$, and ſo on; and $\frac{a}{b}$ is a number having to 1 ſuch a ratio as is equal to the ratio of b to 1 decompounded with, or its inverſe, the ratio of 1 to b compounded with the ratio of a to 1, and, by theorem 2d, is truly expreſſed by $a - a\cdot\frac{b-1}{b}$ or $a + a\cdot\frac{1-b}{b}$; $\frac{a}{bc}$ is a number having to 1 ſuch a ratio as is produced by decompounding the ratios of b to 1 and c to 1, or compounding their inverſe, the ratios of 1 to b and 1 to c, with the ratio of a to 1, and, by the ſame theorem, is equal to $a - a\cdot\frac{b-1}{b} - a\cdot\frac{c-1}{c} + a\cdot\frac{b-1}{b}\cdot\frac{c-1}{c}$, or $a + a\cdot\frac{1-b}{b} + a\cdot\frac{1-c}{c} + a\cdot\frac{1-b}{b}\cdot\frac{1-c}{c}$; $\frac{a}{bcd}$ is a number having to 1 ſuch a ratio as is produced by decompounding the ratios of b to 1, c to 1, and d to 1, or compounding their inverſe, the ratios of 1 to b, 1 to c, and 1 to d, with the ratio of a to 1, and, by the ſame theorem, is equal to $a - a \,.\, \frac{b-1}{b} - a \,.\, \frac{c-1}{c} - a \,.\, \frac{d-1}{d} + a \,.\, \frac{b-1}{b} \,.\, \frac{c-1}{c} + a \,.\, \frac{b-1}{b} \,.\, \frac{d-1}{d} + a \,.\, \frac{c-1}{c} \,.\, \frac{d-1}{d} + a \,.\, \frac{b-1}{b} \,.\, \frac{c-1}{c} \,.\, \frac{d-1}{d}$, or equal to $a + a \,.\, \frac{1-b}{b} + a \,.\, \frac{1-c}{c} + a \,.\, \frac{1-d}{d} + a \,.\, \frac{1-b}{b} \,.\, \frac{1-c}{c} + a \,.\, \frac{1-b}{b} \,.\, \frac{1-d}{d} + a \,.\, \frac{1-c}{c} \,.\, \frac{1-d}{d} + a \,.\, \frac{1-b}{b} \,.\, \frac{1-c}{c} \,.\, \frac{1-d}{d}$, and ſo on. For $2 \times 3 \times 4 \times 5 \times 6$ is the number which is produced by taking a fourth proportional to 1, 2, and 3, a fourth proportional to 1, 4, and $2 \,.\, \frac{3}{1}$, or the firſt fourth proportional, a fourth proportional to 1, 5, and $4 \,.\, \frac{2\,.\,3}{1\,.\,1}$, or the ſecond fourth proportional, a fourth proportional to 1, 6, and $5 \,.\, \frac{4\,.\,2\,.\,3}{1\,.\,1\,.\,1}$ (the number of the operation being always leſs by one than the number of the factors, or numbers multiplied together, or equal to the number of the factors or numbers by which the firſt is multiplied); and $abcdefgh$, &c. is the number which is produced by taking a fourth proportional to 1, b, and a, or $(b \,.\, \frac{a}{1})$, a fourth proportional to 1, c, and $b \,.\, \frac{a}{1}$, or $(c \,.\, \frac{b\,.\,a}{1\,.\,1})$, a fourth proportional to 1, d, and $c \,.\, \frac{b\,.\,a}{1\,.\,1}$, or $(d \,.\, \frac{c\,.\,b\,.\,a}{1\,.\,1\,.\,1})$, a fourth proportional to 1, e, and $d \,.\, \frac{c\,.\,b\,.\,a}{1\,.\,1\,.\,1}$, or $(e \,.\, \frac{d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1}$, a fourth proportional to 1, f, and $e \,.\, \frac{d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1}$, or $(f \,.\, \frac{e\,.\,d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1\,.\,1})$, a fourth proportional to 1, g, and $f \,.\, \frac{e\,.\,d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1\,.\,1}$, or $(g \,.\, \frac{f\,.\,e\,.\,d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1\,.\,1\,.\,1})$, a fourth proportional to 1, h, and $g \,.\, \frac{f\,.\,e\,.\,d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1\,.\,1\,.\,1}$, or $(h \,.\, \frac{g\,.\,f\,.\,e\,.\,d\,.\,c\,.\,b\,.\,a}{1\,.\,1\,.\,1\,.\,1\,.\,1\,.\,1\,.\,1})$, &c. till the number of the operations be equal to the number of the factors b, c, d, e, f, g, h, &c. by which the firſt factor (a) is multiplied. Alſo $\frac{2}{3\,.\,4\,.\,5\,.\,6}$ is the number which is produced by taking a fourth proportional to 3, 1, and 2, or $(1 \,.\, \frac{2}{3})$, a fourth proportional to 4, 1, and $1 \,.\, \frac{2}{3}$, or $(1 \,.\, \frac{1\,.\,2}{3\,.\,4})$, a fourth proportional to 5, 1, and $1 \,.\, \frac{1\,.\,2}{3\,.\,4}$, or $(1 \,.\, \frac{1\,.\,1\,.\,2}{3\,.\,4\,.\,5})$, and a fourth proportional to 6, 1, and $1 \,.\, \frac{1\,.\,1\,.\,2}{3\,.\,4\,.\,5}$, or $(1 \,.\, \frac{1\,.\,1\,.\,1\,.\,2}{3\,.\,4\,.\,5\,.\,6})$, the number of the operations being equal to the number of the factors or numbers 3, 4, 5, and 6, by which the number 2 is divided. And $\frac{a}{bcdefgh, \&c.}$ is the number which is produced by taking a fourth proportional to b, 1, and a, or $(1 . \frac{a}{b})$, a fourth proportional to c, 1, and $1 . \frac{a}{b}$, or $(1 . \frac{1 . a}{c . b})$, a fourth proportional to d, 1, and $1 . \frac{1 . a}{c . b}$, or $(1 . \frac{1 . 1 . a}{d . c . b})$, a fourth proportional to e, 1, and $1 . \frac{1 . 1 . a}{d . c . b}$, or $(1 . \frac{1 . 1 . 1 . a}{e . d . c . b})$, a fourth proportional to f, 1, and $1 . \frac{1 . 1 . 1 . a}{e . d . c . b}$, or $(1 . \frac{1 . 1 . 1 . 1 . a}{f . e . d . c . b})$, a fourth proportional to g, 1, and $1 . \frac{1 . 1 . 1 . 1 . a}{f . e . d . c . b}$, or $(1 . \frac{1 . 1 . 1 . 1 . 1 . a}{g . f . e . d . c . b})$, a fourth proportional to h, 1, and $1 . \frac{1 . 1 . 1 . 1 . 1 . 1 . a}{g . f . e . d . c . b}$, or $(1 . \frac{1 . 1 . 1 . 1 . 1 . 1 . 1 . a}{h . g . f . e . d . c . b})$, &c, till the number of the operations be equal to the number of the factors b, c, d, e, f, g, h, &c, by which, multiplied together, (a) is divided. To dwell longer on illuſtrations would be unneceſſary, even for perſons of the moſt ordinary capacity.—From what has been delivered, however, it manifeſtly appears, that multiplication, diviſion, involution, and evolution, and all ſuch operations in algebra and arithmetic, are, by the principles above laid down, eaſily reducible to, and, when properly and ſcientifically conſidered, in reality nothing elſe, than the augmentation or diminution, the increaſe or decreaſe of arithmetical or numerical magnitudes, by means of proportion and the combination of ratios, when referred to 1, or unit, as the ſtandard of compariſon. But as 1, or unit, combined in any manner, or in any number of ways whatever, ſtill produces unit, this combination of ratios, from the almoſt conſtant diſappearance of the various combinations of unit, and the terms of ratios that are repreſented by it, is not ſo immediately diſcernible in algebra and arithmetic, as in geometry, where the terms and combinations remain entire, without diſappearing, let the ſtandard of compariſon be what it may. This diſappearance, however, of the combinations of unit, in algebra and arithmetic, renders it by far the moſt commodious and convenient ſtandard for common uſe, although the leaſt calculated one poſſible, on the ſaid account, for the purpoſe of ſhewing the connection between the various operations in theſe ſciences, and the real ſcientific principles on which they depend. The famous binomial theorem, given by the illuſtrious Sir Iſaac Newton, was found extremely uſeful for involution and evolution.—But this theorem, as delivered by its author, is clearly an arithmetical one only, having a reference in all its operations to 1, or unit; for it only ſhews the different degrees of increaſe and decreaſe in numerical magnitudes, by the arrangement and combination of ratios, which have 1, or unit, for their conſequents, as well as the ſtandard of compariſon, or, when any numbers are given or aſſumed, what degrees of augmentation or diminution they muſt undergo to have to 1 ſuch ratios as have to the ratios reſpectively of theſe numbers themſelves to 1, the ratios of what are uſually called the indices to 1. Unit, or 1, has therefore very properly, on account of its great convenience and uſe, been commonly uſed in algebra and arithmetic, as the ſtandard of reference for numerical magnitudes, although there ſeems to be no neceſſity for confining our ideas to it, or any other one particular ſtandard; for theſe ſciences, when conſidered generally, as well as geometry, undoubtedly admit of an endleſs variety of ſtandards. The expreſſions delivered in the preceding theorems are not only geometrical, but univerſally metrical; that is, they extend equally to geometry, and all the other abſtract ſciences in general. When they are conſidered, however, in the light in which I have endeavoured to exhibit them, a light, to the beſt of my knowledge, in a great meaſure new; the tranſitions from them, in their geometrical forms, to their algebraic and numerical ones, are ſo natural, ſo ſcientific, and ſo beautiful, that they cannot fail to furniſh the mind with the higheſt pleaſure, ſatisfaction, and delight, in thus pointing out, as it were, at one general view, the connection between thoſe different ſciences, and unfolding the reaſons of their various operations, from the ſame indiſputable and mathematical principles.

THEOREM IV.

Let the ratios of C to D, E to F, G to H, &c, have reſpectively, as in theorem 3d, to the ratio of A to B, the ratios of I to K, L to M, N to O, &c; and let Z be equal to $P + P \cdot \frac{I}{K} + P \cdot \frac{L}{M} + P \cdot \frac{N}{O} + \&c, - P$, or equal to $R - 2P$. Then,

$$\text{1ſt,}\quad B + \frac{Z}{Q} \cdot B \cdot \frac{B-A}{A} + \frac{Z}{Q} \cdot \frac{Z-Q}{2Q} \cdot B \cdot \left.\overline{\frac{B-A}{A}}\right|^2 + \frac{Z}{Q} \cdot \frac{Z-Q}{2Q} \cdot \frac{Z-2Q}{3Q} \cdot B \cdot \left.\overline{\frac{B-A}{A}}\right|^3 + \&c.\ \&c.$$

$$\text{2d, or}\quad B - \frac{Z}{Q} \cdot B \cdot \frac{A-B}{A} + \frac{Z}{Q} \cdot \frac{Z-Q}{2Q} \cdot B \cdot \left.\overline{\frac{A-B}{A}}\right|^2 - \frac{Z}{Q} \cdot \frac{Z-Q}{2Q} \cdot \frac{Z-2Q}{3Q} \cdot B \cdot \left.\overline{\frac{A-B}{A}}\right|^3 + - \&c.\ \&c.$$

$$\text{3d, or}\quad B + \frac{Z}{Q} \cdot C \cdot \frac{B-A}{A} + \frac{Z-Q}{2Q} \cdot D \cdot \frac{B-A}{A} + \frac{Z-2Q}{3Q} \cdot E \cdot \frac{B-A}{A} + \&c.\ \&c.$$

$$\text{4th, or}\quad B - \frac{Z}{Q} \cdot C \cdot \frac{A-B}{A} + \frac{Z-Q}{2Q} \cdot D \cdot \frac{A-B}{A} - \frac{Z-2Q}{3Q} \cdot E \cdot \frac{A-B}{A} + - \&c.\ \&c.$$

ſuppoſing C, D, E, &c, to repreſent the 1ſt, 2d, 3d, &c, terms;

5th, or $A + \frac{Z+Q}{Q}\cdot A\cdot\frac{B-A}{A} + \frac{Z+Q}{Q}\cdot\frac{Z}{2Q}\cdot A\cdot\left.\overline{\frac{B-A}{A}}\right|^{2} + \frac{Z+Q}{Q}\cdot\frac{Z}{2Q}\cdot\frac{Z-Q}{3Q}\cdot A\cdot\left.\overline{\frac{B-A}{A}}\right|^{3} + \&c.\ \&c.$

6th, or $A - \frac{Z+Q}{Q}\cdot A\cdot\frac{A-B}{A} + \frac{Z+Q}{Q}\cdot\frac{Z}{2Q}\cdot A\cdot\left.\overline{\frac{A-B}{A}}\right|^{2} - \frac{Z+Q}{Q}\cdot\frac{Z}{2Q}\cdot\frac{Z-Q}{3Q}\cdot A\cdot\left.\overline{\frac{A-B}{A}}\right|^{3} + - \&c.\ \&c.$

7th, or $A + \frac{Z+Q}{Q}\cdot I\cdot\frac{B-A}{A} + \frac{Z}{2Q}\cdot K\cdot\frac{B-A}{A} + \frac{Z-Q}{3Q}\cdot L\cdot\frac{B-A}{A} + \&c.\ \&c.$

8th, or $A - \frac{Z+Q}{Q}\cdot I\cdot\frac{A-B}{A} + \frac{Z}{2Q}\cdot K\cdot\frac{A-B}{A} - \frac{Z-Q}{3Q}\cdot L\cdot\frac{A-B}{A} + - \&c.\ \&c.$

ſuppoſing I, K, L, &c, to denote the 1ſt, 2d, 3d, &c. terms;

9th, or $\dfrac{B^{\frac{Z}{Q}} + \frac{Z}{Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{B-A}{A} + \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot B^{\frac{Z}{Q}}\cdot\left.\overline{\frac{B-A}{A}}\right|^{2} + \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot\frac{Z-2Q}{3Q}\cdot B^{\frac{Z}{Q}}\cdot\left.\overline{\frac{B-A}{A}}\right|^{3} + \&c.}{B^{\frac{Z-Q}{Q}}}$

10th, or $\dfrac{B^{\frac{Z}{Q}} - \frac{Z}{Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{A-B}{A} + \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot B^{\frac{Z}{Q}}\cdot\left.\overline{\frac{A-B}{A}}\right|^{2} - \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot\frac{Z-2Q}{3Q}\cdot B^{\frac{Z}{Q}}\cdot\left.\overline{\frac{A-B}{A}}\right|^{3} + - \&c.}{B^{\frac{Z-Q}{Q}}}$

11th, or $\dfrac{B^{\frac{Z}{Q}} + \frac{Z}{Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{\Lambda}{A} + \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{\Lambda^{2}}{A^{2}} + \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot\frac{Z-2Q}{3Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{\Lambda^{3}}{A^{3}} + \&c.}{B^{\frac{Z-Q}{Q}}}$

12th, or $\dfrac{B^{\frac{Z}{Q}} - \frac{Z}{Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{\Gamma}{A} + \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{\Gamma^{2}}{A^{2}} - \frac{Z}{Q}\cdot\frac{Z-Q}{2Q}\cdot\frac{Z-2Q}{3Q}\cdot B^{\frac{Z}{Q}}\cdot\frac{\Gamma^{3}}{A^{3}} + - \&c.\ \&c.}{B^{\frac{Z-Q}{Q}}}$

13th, or $\dfrac{B^{\frac{Z-Q}{Q}}}{B^{\frac{Z-Q}{Q}}} \times \left(B + \frac{Z}{Q}\cdot C\cdot\frac{B-A}{A} + \frac{Z-Q}{2Q}\cdot D\cdot\frac{B-A}{A} + \frac{Z-2Q}{3Q}\cdot E\cdot\frac{B-A}{A} + \&c.\ \&c.\right)$

or &c. &c.

14th, or $\dfrac{A^{\frac{Z}{Q}}}{A^{\frac{Z}{Q}}} \times \left(A + \frac{Z+Q}{Q}\cdot I\cdot\frac{B-A}{A} + \frac{Z}{2Q}\cdot K\cdot\frac{B-A}{A} + \frac{Z-Q}{3Q}\cdot L\cdot\frac{B-A}{A} + \&c.\ \&c.\right)$,

or &c. &c. either of theſe two, as well as each of the four preceding ones, producing a geometrical theorem, which, in one particular caſe, viz. when it becomes arithmetical, and is referred to 1, or unit, as the ſtandard, gives the binomial theorem.

15th, or $$\frac{V^{\frac{Z+Q}{Q}} + \frac{Z+Q}{Q}.V^{\frac{Z}{Q}}.W + \frac{Z+Q}{Q}.\frac{Z}{2Q}.V^{\frac{Z-Q}{Q}}.W^{2} + \frac{Z+Q}{Q}.\frac{Z}{2Q}.\frac{Z-Q}{3Q}.V^{\frac{Z-2Q}{Q}}.W^{3} + \&c.}{A^{\frac{Z}{Q}}}$$

ſuppoſing $V + W = B$

16th, or $$\frac{M^{\frac{Z+Q}{Q}} + \frac{Z+Q}{Q}.M^{\frac{Z}{Q}}.N + \frac{Z+Q}{Q}.\frac{Z}{2Q}.M^{\frac{Z-Q}{Q}}.N^{2} - \frac{Z+Q}{Q}.\frac{Z}{2Q}.\frac{Z-Q}{3Q}.M^{\frac{Z-2Q}{Q}}.N^{3} + - \&c.}{A^{\frac{Z}{Q}}}$$

ſuppoſing $M - N = B$;

17th, or $$\overline{\tfrac{S}{T}}\Big|^{\frac{Z+Q}{Q}} \times \left(\frac{T}{S}.B + \frac{Z}{Q}.\frac{T}{S}.E.\frac{\frac{T}{S}.B-A}{A} + \frac{Z}{Q}.\frac{Z-Q}{2Q}.\frac{T}{S}.B.\overline{\frac{\frac{T}{S}.B-A}{A}}\Big|^{2} + \&c.\ \&c.\right),$$

or &c.

18th, or $$\overline{\tfrac{S}{T}}\Big|^{\frac{Z+Q}{Q}} \times \left(A + \frac{Z+Q}{Q}.A.\frac{\frac{T}{S}.B-A}{A} + \frac{Z+Q}{Q}.\frac{Z}{2Q}.A.\overline{\frac{\frac{T}{S}.B-A}{A}}\Big|^{2} + \frac{Z+Q}{Q}.\frac{Z}{2Q}.\frac{Z-Q}{3Q}.A.\overline{\frac{\frac{T}{S}.B-A}{A}}\Big|^{3} + \&c.\right),$$

or &c.

or &c.

has to B ſuch a ratio as has to the ratio ariſing from the decompounding of the ratios of C to D, E to F, G to H, &c. or the compounding of their inverſe, the ratios of D to C, F to E, H to G, &c, with the ratio of A to B, the ratio of P to Q, or has to B ſuch a ratio as has to the ratio of B to A, the ratio of Z to Q, or has to A ſuch a ratio as has to the ratio of B to A, the ratio of Z + Q to Q.

SCHOLIUM.

The equality of theſe formulæ will occaſionally furniſh a great variety of very curious and uſeful theorems and corollaries. If for Z there be put Z − Q, in them, we ſhall have expreſſions thence ariſing that have to A the ſame ratio which theſe formulæ have to B. If the ratio of B to any other magnitude whatever M, of the ſame kind with A and B, be compounded with the ratio which any one of theſe formulæ has to B, expreſſions will thence ariſe, which have to that magnitude the ſame ratio that each of theſe formulæ have to the ſame. If in theſe formulæ, when reduced to the expreſſions juſt now pointed out, which have to A the ſame ratio that they have to B, for B there be ſubſtituted A, and for A, B, we ſhall get expreſſions, to which B has the ſame ratio that they have to B; a circumſtance which will likewiſe furniſh occaſionally an endleſs number of uſeful theorems and corollaries. It may not, perhaps, be unneceſſary to obſerve, that in each of the formulæ in theſe theorems, which, when conſidered as numerical, and referred to 1, or unit, as the ſtandard of compariſon, produces the binomial theorem, it is by no means neceſſary that either A or B be divided into two magnitudes only, of the ſame kind with themſelves: for the ſame reaſoning holds good when A or B is a trinomial magnitude, or equal to any three magnitudes of the ſame kind; when either of them is a quadrinomial magnitude, or equal to any four magnitudes of the ſame kind; when either of them is a quintinomial magnitude, or equal to any five magnitudes of the ſame kind; and ſo on indefinitely. This binomial theorem, however, which the illuſtrious Sir Iſaac Newton derived from induction, and by no means from geometrical reaſoning, is only an arithmetical one, and in ſhort is nothing elſe than a particular caſe of any one of the geometrical theorems delivered above, when ſuppoſed to become numerical, or to be referred to 1, or unit, as the ſtandard of compariſon. But as the human mind is no leſs apt to judge in extremes, than to be in them, the admiring world readily aſcribed, as an invention, to Newton, whatever he happened to deliver. This has certainly been the caſe with the binomial theorem, which furniſhed him with the greateſt aſſiſtance on moſt occaſions, and which, however much it might be an invention on his part, without his knowing that it had been invented ſooner (a circumſtance I am naturally inclined to believe), was certainly diſcovered before he thought of it, as an ingenious and learned friend of mine (Dr. Hutton) has clearly and indiſputably made appear, in the Introduction to his Mathematical Tables *, which have long been in the preſs, and are ſoon expected to be publiſhed.

THEOREM V.

Let there be any number (n) of ratios, viz. A : B, C : D, E : F, G : H, &c. Then the magnitude which has to B the ratio of A to B, together with the magnitude which has to B the ratio that is produced by compounding the ratio of C to D with the ratio of A to B, together with the magnitude, which has to B the ratio that is produced by compounding the ratios of C to D and E to F with the ratio of A to B, and ſo on for (n) terms to the laſt, or

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* Theſe Tables have been publiſhed ſince this was written.

$A + \frac{A \cdot C}{D} + \frac{A \cdot C \cdot E}{D \cdot F} + \frac{A \cdot C \cdot E \cdot G}{D \cdot F \cdot H}$ + &c. to (n) terms is equal to

$\frac{n}{1} \cdot A$, $+ \frac{n-1}{1} \cdot A \cdot \frac{C-D}{D} + \frac{n-2}{1} \cdot A \cdot \frac{E-F}{F} + \frac{n-3}{1} \cdot A \cdot \frac{G-H}{H}$ + &c. to ($n-1$) terms,

$+ \frac{n-2}{1} \cdot A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} + \frac{n-3}{1} \cdot A \cdot \frac{C-D}{D} \cdot \frac{G-H}{H}$ + &c. to ($n-2$) terms,

$+ \frac{n-3}{1} \cdot A \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ + &c. to ($n-3$) terms, + &c. &c. to (1) term,

$+ \frac{n-3}{1} \cdot A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ + &c. to ($n-3$) terms + &c. &c. to (1) term, + &c. &c.

or $\frac{n}{1} \cdot B$, $+ \frac{n}{1} \cdot B \cdot \frac{A-B}{B} + \frac{n-1}{1} \cdot B \cdot \frac{C-D}{D} + \frac{n-2}{1} \cdot B \cdot \frac{E-F}{F} + \frac{n-3}{1} \cdot B \cdot \frac{G-H}{H}$ + &c. to (n) terms,

$+ \frac{n-1}{1} \cdot B \cdot \frac{A-B}{B} \cdot \frac{C-D}{D} + \frac{n-2}{1} \cdot B \cdot \frac{A-B}{B} \cdot \frac{E-F}{F}$ + &c. to ($n-1$) terms,

$+ \frac{n-2}{1} \cdot B \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} + \frac{n-3}{1} \cdot B \cdot \frac{C-D}{D} \cdot \frac{G-H}{H}$ + &c. to ($n-2$) terms,

$+ \frac{n-3}{1} \cdot B \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ + &c. to ($n-3$) terms + &c. &c. to (1) term,

$+ \frac{n-2}{1} \cdot B \cdot \frac{A-B}{B} \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} + \frac{n-3}{1} \cdot B \cdot \frac{A-B}{B} \cdot \frac{C-D}{D} \cdot \frac{G-H}{H}$ + &c. to ($n-2$) terms,

$+ \frac{n-3}{1} \cdot B \cdot \frac{A-B}{B} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ + &c. to ($n-3$) terms + &c. &c. to (1) term,

$+ \frac{n-3}{1} \cdot B \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ + &c. to ($n-3$) terms + &c. &c. to (1) term,

$+ \frac{n-3}{1} \cdot B \cdot \frac{A-B}{B} \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ + &c. to ($n-3$) terms + &c. &c. to (1) term, + &c. &c.

wherein the number of combinations of $\frac{C-D}{D}$, $\frac{E-F}{F}$, $\frac{G-H}{H}$, &c, taken one by one, with A, is $\frac{n}{1} \cdot \frac{n-1}{2}$, taken two by two with A, is $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, taken three by three with A, is $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$, and ſo on, and the number of combinations of the ſame, viz. $\frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$, &c, taken one by one with B, is $\frac{n+1}{1} \cdot \frac{n}{2}$, taken two by two with B, is $\frac{n+1}{1} \cdot \frac{n}{2} \cdot \frac{n-1}{3}$, taken three by three with B, is $\frac{n+1}{1} \cdot \frac{n}{2} \cdot \frac{n-1}{3} \cdot \frac{n-2}{4}$, and ſo on (theſe numbers, by theorem 1ſt, correſponding reſpectively to the next ſuperior orders of ſuch combinations, in a magnitude, which is produced by compounding the ratios of C to D, E to F, G to H, &c, continued to the number (n) with the ratio of A to B).

SCHOLIUM.

If for $\frac{C-D}{D}$, $\frac{E-F}{F}$, $\frac{G-H}{H}$, &c, in theſe expreſſions, there be ſubſtituted $\frac{D-C}{C}$, $\frac{F-E}{E}$, $\frac{H-G}{G}$, &c, reſpectively, they will become equal to $A + A \cdot \frac{D}{C} + A \cdot \frac{D \cdot F}{C \cdot E} + A \cdot \frac{D \cdot F \cdot H}{C \cdot E \cdot G} +$ &c. to (n) terms, or equal to A, together with the magnitude which is produced by decompounding the ratio of C to D, or compounding its inverſe, the ratio of D to C, with the ratio of A to B, together with the magnitude which is produced by decompounding the ratios of C to D and E to F, or compounding their inverſe, the ratios of D to C and F to E, with the ratio of A to B, and ſo on, for (n) terms.—From this theorem may be derived a great variety of new ones, as well as numberleſs inferences, almoſt of the moſt extenſive uſe in the management of many mathematical inveſtigations. Each of the formulæ in this theorem is alſo equal to $\frac{n}{1} \times (A + \frac{n-1}{2} \cdot A \cdot \frac{C-D}{D} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot A \cdot \frac{CE-2CF+DF}{DF} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot A \cdot \frac{CEG-3CEH+3CFH-DFH}{DFH} +$ &c. $(\frac{1}{n} \times n$th, or laſt term $- \frac{n-1}{1} \cdot \overline{n-1}$th term $+ \frac{n-1}{1} \cdot \frac{n-2}{2} \cdot \overline{n-2}$th term $- +$ &c. $\pm A))$; in which all the terms after the one immediately preceding that which vaniſhes, or becomes equal to nothing, will occaſionally be left out, or diſappear. For this is the expreſſion in its perfect form, when each of its terms continued to the number (n) is equal to ſomething. This expreſſion is, truly and ſtrictly ſpeaking, geometrical; and in that particular caſe when it is ſuppoſed to become arithmetical, it gives Sir Iſaac Newton's theorem for the ſummation of ſeries of numbers by means of their differences. The formulæ in the theorem itſelf will, however, be ſometimes found even more convenient than this.—And preciſely in the ſame manner with thoſe formulæ, and this expreſſion, may the aggregates of the firſt, ſecond, third, fourth, &c. differences of theſe magnitudes, and the aggregates of the firſt, ſecond, third, &c. differences of theſe differences, and ſo on, be repreſented.

THEOREM VI.

Let A have to B, a magnitude of the ſame kind with itſelf, any ratio whatever; then A together with the magnitude which has to B the duplicate ratio of A to B; together with the magnitude which has to B the triplicate ratio of A to B; together with the magnitude which has to B the quadruplicate ratio of A to B, and ſo on, for any number of terms (*n*),

or $A + A.\frac{A}{B} + A.\frac{A^2}{B^2} + A.\frac{A^3}{B^3} + A.\frac{A^4}{B^4} + \&c.\ A.\frac{A^{n-1}}{B^{n-1}}$

is equal to each of the following expreſſions:

1ſt, $$\frac{n}{1}\times\left(A+\frac{n-1}{2}.A.\frac{A-B}{B}+\frac{n-1}{2}.\frac{n-2}{3}.A.\overline{\left.\frac{A-B}{B}\right|}^{2}+\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.A.\overline{\left.\frac{A-B}{B}\right|}^{3}+\&c.\ \frac{1}{n}.A.\overline{\left.\frac{A-B}{B}\right|}^{n-1}\right),$$

2d, or $$\frac{n}{1}\times\left(B+\frac{n+1}{2}.B.\frac{A-B}{B}+\frac{n+1}{2}.\frac{n-1}{3}.B.\overline{\left.\frac{A-B}{B}\right|}^{2}+\frac{n+1}{2}.\frac{n-1}{3}.\frac{n-2}{4}.B.\overline{\left.\frac{A-B}{B}\right|}^{3}+\&c.\ \frac{1}{n+1}.B.\overline{\left.\frac{A-B}{B}\right|}^{n}\right),$$

3d, or $$\frac{n}{1}\times\frac{B^{n}+\frac{n+1}{2}.B^{n-1}.D+\frac{n+1}{2}.\frac{n-1}{3}.B^{n-2}.D^{2}+\frac{n+1}{2}.\frac{n-1}{3}.\frac{n-2}{4}.B^{n-3}.D^{3}+\&c.\ \frac{1}{n+1}.D^{n}}{B^{n-1}}$$

ſuppoſing $D = A - B$;

4th, or $$\frac{n+1}{1}\times\frac{B^{n}+\frac{n}{2}.B^{n-1}.D+\frac{n}{2}.\frac{n-1}{3}.B^{n-2}.D^{2}+\frac{n}{2}.\frac{n-1}{3}.\frac{n-2}{4}.B^{n-3}.D^{3}+\&c.\ \frac{1}{n+1}.D^{n}}{B^{n-1}}-B,$$

or &c. &c.

In like manner, $A + A.\frac{A^{\frac{P}{Q}}}{B^{\frac{P}{Q}}} + A.\frac{A^{\frac{2P}{Q}}}{B^{\frac{2P}{Q}}} + A.\frac{A^{\frac{3P}{Q}}}{B^{\frac{3P}{Q}}} + \&c.\ A.\frac{A^{\frac{\overline{n-1}.P}{Q}}}{B^{\frac{\overline{n-1}.P}{Q}}}$, is equal to

$$\frac{n}{1}\times\left(A+\frac{n-1}{2}.A.\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}+\frac{n-1}{2}.\frac{n-2}{3}.A.\overline{\left.\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}\right|}^{2}+\&c.\ \frac{1}{n}.A.\overline{\left.\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}\right|}^{n-1}\right),$$

or &c, &c. ſuppoſing P and Q to be any magnitudes whatever, and (*n*) any number whatever, even or odd.

SCHOLIUM.

In like manner $A + A.\frac{C}{D} + A.\frac{C^2}{D^2} + A.\frac{C^3}{D^3} + \&c.\ A.\frac{C^{n-1}}{D^{n-1}}$ is equal to

$$\frac{n}{1} \times \left(A + \frac{n-1}{2}.A.\frac{C-D}{D} + \frac{n-1}{2}.\frac{n-2}{3}.A.\overline{\frac{C-D}{D}}\Big|^{2} + \frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.A.\overline{\frac{C-D}{D}}\Big|^{3} + \&c.\ \frac{1}{n}.A.\overline{\frac{C-D}{D}}\Big|^{n-1}\right),$$

or &c. &c.; and $C + C.\frac{A}{B} + C.\frac{A^2}{B^2} + C.\frac{A^3}{B^3} + \&c.\ C.\frac{A^{n-1}}{B^{n-1}}$ is equal alſo to

$$\frac{n}{1} \times \left(C + \frac{n-1}{2}.C.\frac{A-B}{B} + \frac{n-1}{2}.\frac{n-2}{3}.C.\overline{\frac{A-B}{B}}\Big|^{2} + \frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.C.\overline{\frac{A-B}{B}}\Big|^{3} + \&c.\ \frac{1}{n}.C.\overline{\frac{A-B}{B}}\Big|^{n-1}\right),$$

or &c. &c. and ſo forth. It is evident, from theorem 3d, that each of the expreſſions in this theorem, repreſenting the aggregate of all the magnitudes of $A + A.\frac{A}{B} + A.\frac{A^2}{B^2} + \&c.\ A.\frac{A^{n-1}}{B^{n-1}}$ is equal to $\left(\frac{A^{n+1}}{B^{n}} - A\right) \div \frac{A-B}{B}$, or is a fourth proportional to the difference of A and B, B, and the difference of the $\overline{n+1}^{\text{th}}$ term and the firſt A. It is likewiſe manifeſt, that the average magnitude, or that of which a multiple by (n), the number of terms, is equal to the aggregate of the whole, is truly expreſſed by

$$A + \frac{n-1}{1}.\frac{1}{2}.A.\frac{A-B}{B} + \frac{n-1}{1}.\frac{n-2}{2}.\frac{1}{3}.A.\overline{\frac{A-B}{B}}\Big|^{2} + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}.\frac{1}{4}.A.\overline{\frac{A-B}{B}}\Big|^{3} + \&c.\ \frac{1}{n}.A.\overline{\frac{A-B}{B}}\Big|^{n-1},$$

or by the magnitude, which has to B the (n^{th}) ratio of A to B, with the co-efficients of its component terms, reckoned from the 1ſt incluſively, divided by the natural numbers 1, 2, 3, 4, 5, 6, &c. reſpectively; and that the magnitude, of which a multiple by ($n+1$) is equal to the ſaid aggregate, is truly expreſſed by

$$\frac{B^{n} + \frac{n}{1}.\frac{1}{2}.B^{n-1}.D + \frac{n}{1}.\frac{n-1}{2}.\frac{1}{3}.B^{n-2}.D^2 + \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{1}{4}.B^{n-3}.D^3 + \&c.\ \frac{1}{n+1}.D^{n}}{B^{n-1}} - \frac{1}{n+1}.B,$$

or the magnitude, by which $\frac{1}{n+1}.B$ differs from the geometrical binomial $\frac{\overline{B+D}|^{n}}{B^{n-1}}$, (which has to B the ($n^{\text{th}}$) ratio of A to B), with the co-efficients of its component terms, reckoned from the 1ſt incluſively, divided by the natural numbers 1, 2, 3, 4, 5, 6, 7, &c. reſpectively, and ſo on. That many very uſeful propoſitions and inferences may likewiſe be drawn from this theorem, it is almoſt needleſs for me to obſerve. It is alſo evident, that each of the expreſſions in this theorem, for

$A + A\,.\,\frac{A}{B} + A\,.\,\frac{A^2}{B^2} +$ &c. to (n) terms is equal likewiſe to each of the following ones:

$$\frac{n}{1} \times \frac{A + \frac{nr-1}{2}\,.\,A\,.\,\frac{A-B}{B} + \frac{nr-1}{2}\,.\,\frac{nr-2}{3}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{2} + \frac{nr-1}{2}\,.\,\frac{nr-2}{3}\,.\,\frac{nr-3}{4}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{3} + \&c.\ \frac{1}{nr}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{nr-1}}{1 + \frac{r-1}{2}\,.\,\frac{A^n - B^n}{B^n} + \frac{r-1}{2}\,.\,\frac{r-2}{3}\,.\,\overline{\frac{A^n - B^n}{B^n}}\Big|^{2} + \frac{r-1}{2}\,.\,\frac{r-2}{3}\,.\,\frac{r-3}{4}\,.\,\overline{\frac{A^n - B^n}{B^n}}\Big|^{3} + \&c.\ \frac{1}{r}\,.\,\overline{\frac{A^n - B^n}{B^n}}\Big|^{r-1}}$$

$$\text{or } \frac{n}{1} \times \frac{B + \frac{nr+1}{2}\,.\,B\,.\,\frac{A-B}{B} + \frac{nr+1}{2}\,.\,\frac{nr-1}{3}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{2} + \frac{nr+1}{2}\,.\,\frac{nr-1}{3}\,.\,\frac{nr-2}{4}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{3} + \&c.\ \frac{1}{nr+1}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{nr}}{1 + \frac{r-1}{2}\,.\,\frac{A^n - B^n}{B^n} + \frac{r-1}{2}\,.\,\frac{r-2}{3}\,.\,\overline{\frac{A^n - B^n}{B^n}}\Big|^{2} + \frac{r-1}{2}\,.\,\frac{r-2}{3}\,.\,\frac{r-3}{4}\,.\,\overline{\frac{A^n - B^n}{B^n}}\Big|^{3} + \&c.\ \frac{1}{r}\,.\,\overline{\frac{A^n - B^n}{B^n}}\Big|^{r-1}}$$

or &c. &c. (r) being any number whatever, either odd or even.

THEOREM VI.

Let (m) denote any number, and (n) any number whatever, odd or even; then the geometrical expreſſion $A - A\,.\,\frac{A}{B} + A\,.\,\frac{A^2}{B^2} - A\,.\,\frac{A^3}{B^3} + -$ &c. $- A\,.\,\frac{A^{m-1}}{B^{m-1}}$ is equal to the following, viz.

1ſt, $m\,.\,\frac{B-A}{B+A} \times \Big(A + \frac{m-1}{2}\,.\,A\,.\,\frac{A-B}{B} + \frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{2} + \&c.\ \frac{1}{m}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{m-1}\Big)$,

2d, or $m\,.\,\frac{B-A}{B+A} \times \Big(B + \frac{m+1}{2}\,.\,B\,.\,\frac{A-B}{B} + \frac{m+1}{2}\,.\,\frac{m-1}{3}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{2} + \&c.\ \frac{1}{m+1}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{m}\Big)$,

3d, or $\frac{m+1}{1}\,.\,\frac{B-A}{B+A} \times \Big(B + \frac{m}{1}\,.\,\frac{1}{2}\,.\,B\,.\,\frac{A-B}{B} + \frac{m}{1}\,.\,\frac{m-1}{2}\,.\,\frac{1}{3}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{2} + \&c.\ \frac{1}{m+1}\,.\,B\,.\,\overline{\frac{A-B}{B}}\Big|^{m}\Big) - B\,.\,\frac{B-A}{B+A}$

4th, or $\frac{m+1}{1}\,.\,\frac{B-A}{B+A} \times \frac{B^m + \frac{m}{1}\,.\,\frac{1}{2}\,.\,B^{m-1}\,.\,D + \frac{m}{1}\,.\,\frac{m-1}{2}\,.\,\frac{1}{3}\,.\,B^{m-2}\,.\,D^2 + \&c.\ \frac{1}{m+1}\,.\,D^m}{B^{m-1}} - B\,.\,\frac{B-A}{B+A}$,

if $D = A - B$,

5th, or is $= \frac{m}{1}\,.\,\frac{B-A}{B+A} \times \frac{A + \frac{mn-1}{2}\,.\,A\,.\,\frac{A-B}{B} + \frac{mn-1}{2}\,.\,\frac{mn-2}{3}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{2} + \&c.\ \frac{1}{mn}\,.\,A\,.\,\overline{\frac{A-B}{B}}\Big|^{mn-1}}{1 + \frac{n-1}{2}\,.\,\frac{A^m - B^m}{B^m} + \frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,\overline{\frac{A^m - B^m}{B^m}}\Big|^{2} + \&c.\ \frac{1}{n}\,.\,\overline{\frac{A^m - B^m}{B^m}}\Big|^{n-1}}$,

or &c. &c.

In like manner $A - A\,.\,\frac{A^{\frac{P}{Q}}}{B^{\frac{P}{Q}}} + A\,.\,\frac{A^{\frac{2P}{Q}}}{B^{\frac{2P}{Q}}} - A\,.\,\frac{A^{\frac{3P}{Q}}}{B^{\frac{3P}{Q}}} + -$ &c. $- A\,.\,\frac{A^{\frac{\overline{m-1}\,.\,P}{Q}}}{B^{\frac{\overline{m-1}\,.\,P}{Q}}}$, is expreſſed by

$$m\cdot\frac{B^{\frac{P}{Q}}-A^{\frac{P}{Q}}}{B^{\frac{P}{Q}}+A^{\frac{P}{Q}}}\times\left(A+\frac{m-1}{2}\cdot A\cdot\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}+\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot A\cdot\left.\overline{\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}}\right|^{2}+\&c.\ \frac{1}{m}\cdot A\cdot\left.\overline{\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}}\right|^{m-1}\right),$$

or &c. &c. ſuppoſing P and Q to be any two magnitudes whatever.

Cor. 1. From this and theorem 6th, it plainly appears, that any ſuch geometrical expreſſion as $A+A\cdot\frac{A^2}{B^2}+A\cdot\frac{A^4}{B^4}+\&c.\ A\cdot\frac{A^{m-2}}{B^{m-2}}$ (m being any even number) is equal to

$$\frac{m}{1}\cdot\frac{B}{B+A}\times\left(A+\frac{m-1}{2}\cdot A\cdot\frac{A-B}{B}+\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot A\cdot\left.\overline{\frac{A-B}{B}}\right|^{2}+\&c.\ \frac{1}{m}\cdot A\cdot\left.\overline{\frac{A-B}{B}}\right|^{m-1}\right),\ \text{or}$$

$$\frac{m}{1}\cdot\frac{B}{B+A}\times\frac{B^{m}+\frac{m+1}{2}\cdot B^{m-1}\cdot D+\frac{m+1}{2}\cdot\frac{m-1}{3}\cdot B^{m-2}\cdot D^{2}+\&c.\ \frac{1}{m+1}\cdot D^{m}}{B^{m-1}};$$

and that any ſuch geometrical expreſſion as $A+A\cdot\frac{C^2}{D^2}+A\cdot\frac{C^4}{D^4}+\&c.\ A\cdot\frac{C^{m-2}}{D^{m-2}}$ is equal to

$$\frac{m}{1}\cdot\frac{D}{D+C}\times\left(A+\frac{m-1}{2}\cdot A\cdot\frac{C-D}{D}+\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot A\cdot\left.\overline{\frac{C-D}{D}}\right|^{2}+\&c.\ \frac{1}{m}\cdot A\cdot\left.\overline{\frac{C-D}{D}}\right|^{m-1}\right);$$

and ſo on.

Cor. 2. It is likewiſe evident, that any ſuch geometrical expreſſion as $A\cdot\frac{A}{B}+A\cdot\frac{A^3}{B^3}+A\cdot\frac{A^5}{B^5}+\&c.\ A\cdot\frac{A^{m-1}}{B^{m-1}}$ (m being any even number) is equal to

$$\frac{m}{1}\cdot\frac{A}{B+A}\times\left(A+\frac{m-1}{2}\cdot A\cdot\frac{A-B}{B}+\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot A\cdot\left.\overline{\frac{A-B}{B}}\right|^{2}+\&c.\ \frac{1}{m}\cdot A\cdot\left.\overline{\frac{A-B}{B}}\right|^{m-1}\right),\ \text{or}$$

$$\frac{m}{1}\cdot\frac{A}{B+A}\times\frac{B^{m}+\frac{m+1}{2}B^{m-1}\cdot D+\frac{m-1}{2}\cdot\frac{m-1}{3}\cdot B^{m-2}\cdot D^{2}+\&c.\ \frac{1}{m-1}\cdot D^{m}}{B^{m-1}},\ \text{or \&c. \&c.}$$

and that any ſuch expreſſion as $A\cdot\frac{C}{D}+A\cdot\frac{C^3}{D^3}+A\cdot\frac{C^5}{D^5}+\&c.\ A\cdot\frac{C^{m-1}}{D^{m-1}}$ is equal to

$$\frac{m}{1}\cdot\frac{C}{D+C}\times\left(A+\frac{m-1}{2}\cdot A\cdot\frac{C-D}{D}+\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot A\cdot\left.\overline{\frac{C-D}{D}}\right|^{2}+\&c.\ \frac{1}{m}\cdot A\cdot\left.\overline{\frac{C-D}{D}}\right|^{m-1}\right);$$

and ſo on.

Cor. 3. It is alſo manifeſt, that a fourth proportional to A, the expreſſion

$A + A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} + A \cdot \frac{A^3}{B^3} + \&c.\ A \cdot \frac{A^{m-1}}{B^{m-1}}$, and the expreſſion

$A - A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} - A \cdot \frac{A^3}{B^3} + - \&c. - A \cdot \frac{A^{m-1}}{B^{m-1}}$, is equal to

$$\frac{m^2}{A} \cdot \frac{B-A}{B+A} \times \left(A + \frac{m-1}{2} \cdot A \cdot \frac{A-B}{B} + \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot A \cdot \overline{\frac{A-B}{B}}\Big|^{2} + \&c.\ \frac{1}{m} \cdot A \cdot \overline{\frac{A-B}{B}}\Big|^{m-1}\right)^2,$$

$$\text{or } \frac{m^2}{A} \cdot \frac{B-A}{B+A} \times \left(\frac{B^m + \frac{m+1}{2} \cdot B^{m-1} \cdot D + \frac{m-1}{2} \cdot \frac{m-1}{3} \cdot B^{m-2} D^2 + \&c.\ \frac{1}{m+1} \cdot D^m}{B^{m-1}}\right)^2,$$

or &c. &c. and ſo on for other magnitudes.

Cor. 4. It is likewiſe plain, that a fourth proportional to the expreſſion

$A - A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} - A \cdot \frac{A^3}{B^3} + - \&c. - A \cdot \frac{A^{m-1}}{B^{m-1}}$, A, and the expreſſion

$A + A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} + A \cdot \frac{A^3}{B^3} + \&c.\ A \cdot \frac{A^{m-1}}{B^{m-1}}$ is equal to $A \cdot \frac{B+A}{B-A}$, and vice verſa; and ſo on for other magnitudes.

Cor. 5. Whence it is manifeſt, that the expreſſions

$$A + A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} + \&c.\ A \cdot \frac{A^{m-1}}{B^{m-1}},$$

$$A - A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} - A \cdot \frac{A^3}{B^3} + - \&c.\ A \cdot \frac{A^{m-1}}{B^{m-1}},$$

$$A + A \cdot \frac{A^2}{B^2} + A \cdot \frac{A^4}{B^4} + \&c.\ A \cdot \frac{A^{m-2}}{B^{m-2}}, \text{ and}$$

$$A \cdot \frac{A}{B} + A \cdot \frac{A^3}{B^3} + A \cdot \frac{A^5}{B^5} + \&c.\ A \cdot \frac{A^{m-1}}{B^{m-1}},$$

(m being any even number whatever), are to one another as $B + A$, $B - A$, B, and A reſpectively; and that each of the formulæ in this theorem, for the ſecond of theſe expreſſions, is equal to $\frac{A^{m+1}}{B^m} - A \times \frac{B}{A-B} \cdot \frac{B-A}{A+B}$.

THEOREM VIII.

Let (b and d) denote any odd numbers whatever. Then the geometrical expreſſion $A - A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} - A \cdot \frac{A^3}{B^3} + - \&c.\ A \cdot \frac{A^{b-1}}{B^{b-1}}$ is equal to

1ſt, $$2B\cdot\frac{A}{B+A}+b\cdot\frac{A-B}{B+A}\times\left(A+\frac{b-1}{2}\cdot A\cdot\frac{A-B}{B}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{b-1}\right),$$

2d, or $$2B\cdot\frac{A}{B+A}+b\cdot\frac{A-B}{B+A}\times\left(B+\frac{b+1}{2}\cdot B\cdot\frac{A-B}{B}+\frac{b+1}{2}\cdot\frac{b-1}{3}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b+1}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{b}\right)-B\cdot\frac{A-B}{A+B},$$

3d, or $$\left.\begin{aligned}&2B\cdot\frac{A}{B+A}+\frac{b+1}{1}\cdot\frac{A-B}{B+A}\times\left(B+\frac{b}{2}\cdot B\cdot\frac{A-B}{B}+\frac{b}{2}\cdot\frac{b-1}{3}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b+1}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{b}\right)-B\cdot\frac{A-B}{A+B}\\ &=B+\frac{b+1}{1}\cdot\frac{A-B}{A+B}\times\left(B+\frac{b}{2}\cdot B\cdot\frac{A-B}{B}+\frac{b}{2}\cdot\frac{b-1}{3}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b+1}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{b}\right).\end{aligned}\right\}$$

4th, or $$\frac{2B\cdot\frac{A}{B+A}+\frac{bd}{1}\cdot\frac{A-B}{B+A}\times\left(A+\frac{bd-1}{2}\cdot A\cdot\frac{A-B}{B}+\frac{bd-1}{2}\cdot\frac{bd-2}{3}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{bd}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{bd-1}\right)}{\frac{2B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}+A^{\frac{P}{Q}}}+\frac{d}{1}\cdot\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}+A^{\frac{P}{Q}}}\times\left(1+\frac{d-1}{2}\cdot\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}+\frac{d-1}{2}\cdot\frac{d-2}{3}\cdot\overline{\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}}\Big|^{2}+\&c.\ \frac{1}{d}\cdot A\cdot\overline{\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}}\Big|^{d-1}\right)}$$

or &c. &c. ſuppoſing P and Q to be any magnitudes whatever.

In like manner $A-A\cdot\frac{C}{D}+A\cdot\frac{C^{2}}{D^{2}}-A\cdot\frac{C^{3}}{D^{3}}+-\&c.\ A\cdot\frac{C^{b-1}}{D^{b-1}}$ is expreſſed by

$$2D\cdot\frac{A}{D+C}+b\cdot\frac{C-D}{C+D}\times\left(A+\frac{b-1}{2}\cdot A\cdot\frac{C-D}{D}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot A\cdot\overline{\frac{C-D}{D}}\Big|^{2}+\&c.\ \frac{1}{b}\cdot A\cdot\overline{\frac{C-D}{D}}\Big|^{b-1}\right),$$

or &c. &c. and $A^{\frac{P}{Q}}-A^{\frac{P}{Q}}\cdot\frac{A^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}+A^{\frac{P}{Q}}\cdot\frac{A^{\frac{2P}{Q}}}{B^{\frac{2P}{Q}}}-A^{\frac{P}{Q}}\cdot\frac{A^{\frac{3P}{Q}}}{B^{\frac{3P}{Q}}}+-\&c.\ A^{\frac{P}{Q}}\cdot\frac{A^{\frac{\overline{b-1}\cdot P}{Q}}}{B^{\frac{\overline{b-1}\cdot P}{Q}}}$ is equal to

$$2B^{\frac{P}{Q}}\cdot\frac{A^{\frac{P}{Q}}}{A^{\frac{P}{Q}}+B^{\frac{P}{Q}}}+b\cdot\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{A^{\frac{P}{Q}}+B^{\frac{P}{Q}}}\times\left(A^{\frac{P}{Q}}+\frac{b-1}{2}\cdot A^{\frac{P}{Q}}\cdot\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot A^{\frac{P}{Q}}\cdot\overline{\frac{A^{\frac{P}{Q}}-B^{\frac{P}{Q}}}{B^{\frac{P}{Q}}}}\Big|^{2}\cdot+\&c.\ \frac{1}{b}\cdot A^{\frac{P}{Q}}\cdot\frac{A^{\frac{P}{Q}}-}{B^{\frac{P}{Q}}}\right.$$

or &c. &c. and ſo on for other magnitudes.

Cor. 1. $A+A\cdot\frac{A^{2}}{B^{2}}+A\cdot\frac{A^{4}}{B^{4}}+A\cdot\frac{A^{6}}{B^{6}}+\&c.\ A\cdot\frac{A^{b-1}}{B^{b-1}}$ (b being any odd number) is equal to

$$B\cdot\frac{A}{A+B}+b\cdot\frac{A}{A+B}\times\left(A+\frac{b-1}{2}\cdot A\cdot\frac{A-B}{B}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot\frac{b-3}{4}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{3}+\&c.\ \frac{1}{b}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{b-1}\right.$$

or $$\frac{b+1}{1}\cdot\frac{A}{A+B}\times\left(B+\frac{b}{2}\cdot B\cdot\frac{A-B}{B}+\frac{b}{2}\cdot\frac{b-1}{3}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\frac{b}{2}\cdot\frac{b-1}{3}\cdot\frac{b-2}{4}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{3}+\&c.\ \frac{1}{b+1}\cdot B\cdot\overline{\frac{A+B}{B}}\Big|^{b}\right),$$

or &c. &c. and ſo on for other magnitudes.

Cor. 2. $A\cdot\frac{A}{B}+A\cdot\frac{A^{3}}{B^{3}}+A\cdot\frac{A^{5}}{B^{5}}+A\ \frac{A^{7}}{B^{7}}+\&c.\ A\cdot\frac{A^{b-2}}{B^{b-2}}$ is likewiſe equal to

$$-B\cdot\frac{A}{A+B}+b\cdot\frac{B}{A+B}\times\left(A+\frac{b-1}{2}\cdot A\cdot\frac{A-B}{B}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b}\cdot A\cdot\overline{\frac{A-B}{B}}\Big|^{b-1}\right),\ \text{or}$$

$$-B+\frac{b+1}{1}\cdot\frac{A}{A+B}\times\left(B+\frac{b}{2}\cdot B\cdot\frac{A-B}{B}+\frac{b}{2}\cdot\frac{b-1}{3}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\frac{b}{2}\cdot\frac{b-1}{3}\cdot\frac{b-2}{4}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{3}+\&c.\ \frac{1}{b+1}\cdot B\cdot\overline{\frac{A-B}{B}}\Big|^{b}\right)$$

or &c. &c. and ſo on for other magnitudes.

Cor. 3. A fourth proportional to A, $A + A . \frac{A}{B} + A . \frac{A^2}{B^2} + A . \frac{A^3}{B^3} + \&c. \; A . \frac{A^{b-1}}{B^{b-1}}$, and

$$A - A . \frac{A}{B} + A . \frac{A^2}{B^2} - A . \frac{A^3}{B^3} + - \&c. \; A . \frac{A^{b-1}}{B^{b-1}},$$

is in like manner equal to

$$\frac{\overline{b+1}}{A} . \frac{A-B}{A+B} \times (A + \frac{b}{2} . B . \frac{A-B}{B} + \frac{b}{2} . \frac{b-1}{3} . B . \overline{\frac{A-B}{B}}\Big|^{2} + \frac{b}{2} . \frac{b-1}{3} . \frac{b-2}{4} . B . \overline{\frac{A-B}{B}}\Big|^{3} + \&c. \frac{1}{b+1} . B . \overline{\frac{A-B}{B}}\Big|^{b})^{2}$$

$$+ \frac{B}{A} . \frac{2B}{A+B} + \frac{b+1}{1} \times (B + \frac{b}{2} . B . \frac{A-B}{B} + \frac{b}{2} . \frac{b-1}{3} . B . \overline{\frac{A-B}{B}}\Big|^{2} + \frac{b}{2} . \frac{b-1}{3} . \frac{b-2}{4} . B . \overline{\frac{A-B}{B}}\Big|^{3} + \&c. \frac{1}{b+1} . B . \overline{\frac{A-B}{B}}\Big|^{b})$$

$- \frac{B^2}{A}$ or &c. &c. and ſo on for other magnitudes.

Cor. 4. A fourth proportional to $A - A . \frac{A}{B} + A . \frac{A^2}{B^2} - A . \frac{A^3}{B^3} + - \&c. \; A . \frac{A^{b-1}}{B^{b-1}}$, A, and

$$A + A . \frac{A}{B} + A . \frac{A^2}{B^2} + A . \frac{A^3}{B^3} + \&c. \; A . \frac{A^{b-1}}{B^{b-1}},$$

is in like manner equal to

$$. \frac{A+B}{A-B} \times \frac{B + \frac{b}{2} . B . \frac{A-B}{B} + \frac{b}{2} . \frac{b-1}{3} . B . \overline{\frac{A-B}{B}}\Big|^{2} + \frac{b}{2} . \frac{b-1}{3} . \frac{b-2}{4} . B . \overline{\frac{A-B}{B}}\Big|^{3} + \&c. \frac{1}{b+1} . B . \overline{\frac{A-B}{B}}\Big|^{b} - \frac{1}{b+1} . B}{B + \frac{b}{2} . B . \frac{A-B}{B} + \frac{b}{2} . \frac{b-1}{3} . B . \overline{\frac{A-B}{B}}\Big|^{2} + \frac{b}{2} . \frac{b-1}{3} . \frac{b-2}{4} . B . \overline{\frac{A-B}{B}}\Big|^{3} + \&c. \frac{1}{b+1} . B . \overline{\frac{A-B}{B}}\Big|^{b} + \frac{1}{b+1} . B . \frac{A+B}{A-B}},$$

and vice verſa; and ſo on for other magnitudes.

Cor. 5. From the two preceding theorems, and this, it likewiſe appears, that

$\frac{A}{B} + \frac{A^2}{B^2} + \frac{A^3}{B^3} + \frac{A^4}{B^4} + \frac{A^5}{B^5} + \&c. \frac{A^{n-1}}{B^{n-1}}$ (ſuppoſing n to denote any number) is equal to

$$\frac{n-1}{1} + \frac{n}{1} . \frac{n-1}{2} . \frac{A-B}{B} + \frac{n}{1} . \frac{n-1}{2} . \frac{n-2}{3} . \overline{\frac{A-B}{B}}\Big|^{2} + \frac{n}{1} . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . \overline{\frac{A-B}{B}}\Big|^{3} + \&c. \overline{\frac{A-B}{B}}\Big|^{n-1},$$ or &c. &c.

that $1 + \frac{A}{B} + \frac{A^2}{B^2} + \frac{A^3}{B^3} + \frac{A^4}{B^4} + \frac{A^5}{B^5} + \&c. \frac{A^{n-1}}{B^{n-1}}$ is equal to

$$\frac{n}{1} \times (1 + \frac{n-1}{2} . \frac{A-B}{B} + \frac{n-1}{2} . \frac{n-2}{3} . \overline{\frac{A-B}{B}}\Big|^{2} + \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . \overline{\frac{A-B}{B}}\Big|^{3} + \&c. \frac{1}{n} . \overline{\frac{A-B}{B}}\Big|^{n-1}),$$ or &c. &c.

that $1 - \frac{A}{B} + \frac{A^2}{B^2} - \frac{A^3}{B^3} + \frac{A^4}{B^4} - + \&c. - \frac{A^{m-1}}{B^{m-1}}$ (m being any even number) is equal to

$$. \frac{B-A}{A+B} \times (1 + \frac{m-1}{2} . \frac{A-B}{B} + \frac{m-1}{2} . \frac{m-2}{3} . \overline{\frac{A-B}{B}}\Big|^{2} + \&c. \frac{1}{m} . \frac{A^{m-1}}{B^{m-1}}),$$ or &c. &c.

$1 + \frac{A^2}{B^2} + \frac{A^4}{B^4} + \frac{A^6}{B^6} + \&c. \frac{A^{m-2}}{B^{m-2}}$ is $= m . \frac{B}{A+B} \times (1 + \frac{m-1}{2} . \frac{A-B}{B} + \frac{m-1}{2} . \frac{m-2}{3} . \overline{\frac{A-B}{B}}\Big|^{2} + \&c. \frac{1}{m} . \overline{\frac{A-B}{B}}\Big|^{m-1})^{2}$,

c. &c. that $\frac{A}{B} + \frac{A^3}{B^3} + \frac{A^5}{B^5} + \&c. \frac{A^{m-1}}{B^{m-1}}$ is $= m . \frac{A}{A+B} \times (1 + \frac{m-1}{2} . \frac{A-B}{B} + \frac{m-1}{2} . \frac{m-2}{3} . \overline{\frac{A-B}{B}}\Big|^{2} + \&c. \frac{1}{m} . \overline{\frac{A-B}{B}}\Big|^{m-1})$,

&c. &c. that a fourth proportional to 1, $1 + \frac{A}{B} + \frac{A^2}{B^2} + \&c. \frac{A^{m-1}}{B^{m-1}}$, and $1 - \frac{A}{B} + \frac{A^2}{B^2} - + \&c. - \frac{A^{m-1}}{B^{m-1}}$ is equal to

$\frac{m^2}{1}\cdot\frac{B-A}{A+B}\times(1+\frac{m-1}{2}\cdot\frac{A-B}{B}+\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{m}\cdot\overline{\frac{A-B}{B}}\Big|^{m-1})^{2}$, or &c. &c. and that a fourth proportional to $1-\frac{A}{B}+\frac{A^2}{B^2}-+\&c.-\frac{A^{m-1}}{B^{m-1}}$, 1, and $1+\frac{A}{B}+\frac{A^2}{B^2}+\&c.\ \frac{A^{m-1}}{B^{m-1}}$, is equal to $\frac{A+B}{A-B}$; that $1-\frac{A}{B}+\frac{A^2}{B^2}-+\&c.\ \frac{A^{b-1}}{B^{b-1}}$ (*b* being any odd number) is equal to $\frac{2B}{A+B}+b\cdot\frac{A-B}{A+B}\times(1+\frac{b-1}{2}\cdot\frac{A-B}{B}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b}\cdot\overline{\frac{A-B}{B}}\Big|^{b-1})$, or &c. &c. that $1+\frac{A^2}{B^2}+\frac{A^4}{B^4}+\&c.\ \frac{A^{b-1}}{B^{b-1}}$ is $=\frac{B}{A+B}+b\cdot\frac{A}{A+B}\times(1+\frac{b-1}{2}\cdot\frac{A-B}{B}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b}\cdot\overline{\frac{A-B}{B}}\Big|^{b-1})$, or &c. that $\frac{A}{B}+\frac{A^3}{B^3}+\&c.\ \frac{A^{b-2}}{B^{b-2}}$ is $=b\cdot\frac{B}{A+B}\times(1+\frac{b-1}{2}\cdot\frac{A-B}{B}+\frac{b-1}{2}\cdot\frac{b-2}{3}\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\&c.\ \frac{1}{b}\cdot\overline{\frac{A-B}{B}}\Big|^{b-1}-\frac{B}{A+B}$; and ſo on.

Cor. 6. It is alſo evident, that ſuppoſing *n* and *r* to repreſent any numbers whatever, $1+\frac{n-1}{2}\cdot\frac{A-B}{B}+\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\overline{\frac{A-B}{B}}\Big|^{3}+\&c.\ \frac{1}{n}\cdot\overline{\frac{A-B}{B}}\Big|^{n-1}$ multiplied by $1+\frac{r-1}{2}\cdot\frac{A^n-B^n}{B^n}+\frac{r-1}{2}\cdot\frac{r-2}{3}\cdot\overline{\frac{A^n-B^n}{B^n}}\Big|^{2}+\frac{r-1}{2}\cdot\frac{r-2}{3}\cdot\frac{r-3}{4}\cdot\overline{\frac{A^n-B^n}{B^n}}\Big|^{3}+\&c.\ \frac{1}{r}\cdot\overline{\frac{A^n-B^n}{B^n}}\Big|^{r-1}$ is equal to $1+\frac{nr-1}{2}\cdot\frac{A-B}{B}+\frac{nr-1}{2}\cdot\frac{nr-2}{3}\cdot\overline{\frac{A-B}{B}}\Big|^{2}+\frac{nr-1}{2}\cdot\frac{nr-2}{3}\cdot\frac{nr-3}{4}\cdot\overline{\frac{A-B}{B}}\Big|^{3}+\&c.\ \frac{1}{nr}\cdot\overline{\frac{A-B}{B}}\Big|^{nr-1}$, and conſequently that $1+\frac{A}{B}+\frac{A^2}{B^2}+\frac{A^3}{B^3}+\&c.\ \frac{A^{n-1}}{B^{n-1}}$ multiplied by $1+\frac{A^n}{B^n}+\frac{A^{2n}}{B^{2n}}+\frac{A^{3n}}{B^{3n}}+\&c.\ \frac{A^{\overline{r-1}\cdot n}}{B^{\overline{r-1}\cdot n}}$, is equal to $1+\frac{A}{B}+\frac{A^2}{B^2}+\frac{A^3}{B^3}+\frac{A^4}{B^4}+\frac{A^5}{B^5}+\&c.\ \frac{A^{nr-1}}{B^{nr-1}}$.

Almoſt numberleſs other corollaries or inferences might be drawn from a compariſon of theſe theorems.

SCHOLIUM.

In any ſuch ſeries of magnitudes as $A+A\cdot\frac{A}{B}+A\cdot\frac{A^2}{B^2}+A\cdot\frac{A^3}{B^3}+\&c.\ \frac{A^{n-1}}{B^{n-1}}$, (*n* being any number whatever) the ſum or aggregate of their firſt differences is equal to a fourth proportional to B, $A \backsim B$ (or the difference of A and B), and

$A + A \cdot \frac{A}{B} + A \cdot \frac{A^2}{B^2} + \&c.\ A \cdot \frac{A^{n-2}}{B^{n-2}}$, or all the magnitudes in the ſeries except the laſt, that is (by theorem 6) equal to $\frac{A \backsim B}{B} \times \frac{n-1}{1} \times (A + \frac{n-2}{2} \cdot A \cdot \frac{A-B}{B} + \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{2} + \&c.\ \frac{1}{n-1} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{n-2}$, or &c. &c.; that of the ſecond differences is in like manner equal to $\left.\overline{\frac{A \backsim B}{B}}\right|^{2} \times \frac{n-2}{1} \times (A + \frac{n-3}{2} \cdot A \cdot \frac{A-B}{B} + \frac{n-3}{2} \cdot \frac{n-4}{3} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{2} + \&c.\ \frac{1}{n-2} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{n-3})$, or &c. &c.; that of the third is $= \left.\overline{\frac{A \backsim B}{B}}\right|^{3} \times \frac{n-3}{1} \times (A + \frac{n-4}{2} \cdot A \cdot \frac{A-B}{B} + \&c.\ \frac{1}{n-3} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{n-4})$, or &c.; that of the fourth is $= \left.\overline{\frac{A \backsim B}{B}}\right|^{4} \times \frac{n-4}{1} \times (A + \frac{n-5}{2} \cdot A \cdot \frac{A-B}{B} + \&c.\ \frac{1}{n-4} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{n-5})$, or &c. &c.; that of the fifth is $= \left.\overline{\frac{A \backsim B}{B}}\right|^{5} \times \frac{n-5}{1} \times (A + \frac{n-6}{1} \cdot A \cdot \frac{A-B}{B} + \&c.\ \frac{1}{n-5} \cdot A \cdot \left.\overline{\frac{A-B}{B}}\right|^{n-6})$, or &c. &c.; and ſo on to the $\overline{n-1}^{\text{th}}$ differences, of which the aggregate is $\left.\overline{\frac{A \backsim B}{B}}\right|^{n-1} \times A$.

THE multiplicity of buſineſs in an active profeſſion had induced the Author of this Paper to lay it aſide in its preſent form, when he was going to add to it many general and intereſting theorems; and having been juſt ordered abroad, he has thought proper to deliver it in its preſent ſtate, with the aſſurance of completing it hereafter.

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ERRATA.

Note *b* at the number of the line denotes counted from the bottom.

Page.	Line.	Correction.
12.	17.	*For* $\frac{P}{1}$, $\frac{P-1}{1}$, *read* $\frac{P}{1}$, $\frac{P-1}{2}$.
14.	8.	*For* $\frac{D-C}{B}$ and $\frac{F-E}{D}$, *read* $\frac{D-C}{D}$ and $\frac{F-E}{F}$.
15.	18.	*For* ($\frac{P}{1}$ erms, *read* ($\frac{P}{1}$) terms.
29.	4.	*For* $+a\frac{b-1}{b}$, $\frac{c-1}{c}$, $\frac{d-1}{d}$, *read* $-a\frac{b-1}{b}$, $\frac{c-1}{c}$, $\frac{d-1}{d}$.
31.	7*b*.	*For* $P+P\frac{1}{K}+P\frac{L}{M}+P\frac{N}{O}+$ &c., $-P$, *read* $P\frac{1}{K}+P\frac{L}{M}+P\frac{N}{O}+$ &c., $-P$.
37.	2*b*.	*For* $\frac{1}{n}A\frac{\overline{A\frac{P}{Q}-B\frac{P}{Q}}^{n-1}}{B\frac{P}{Q}}$ *read* $\frac{1}{n}A\overline{\frac{A\frac{P}{Q}-B\frac{P}{Q}}{B\frac{P}{Q}}}^{n-1}$
39.		*For* Theorem VI. *read* Theorem VII.
40. and 41.	6*b*. 4.	*For* $\frac{m-1}{2}$, $\frac{m-1}{3}$, B^{m-2}, *read* $\frac{m+1}{2}$, $\frac{m-1}{3}$, B^{m-2}.
42.	6.	*For* $\frac{\overline{A\frac{P}{Q}-B\frac{P}{Q}}^{2}}{B\frac{P}{Q}}$, *read* $\overline{\frac{A\frac{P}{Q}-B\frac{P}{Q}}{B\frac{P}{Q}}}^{2}$, and *for* A, $\overline{\frac{A\frac{P}{Q}-B\frac{P}{Q}}{B\frac{P}{Q}}}^{d-1}$ *read* $\overline{\frac{A\frac{P}{Q}-B\frac{P}{Q}}{B\frac{P}{Q}}}^{d-1}$
43.	3*b*.	At the end of A eraſe 2.
44.	3*b*.	*For* $\frac{A\,n-1}{B\,n-1}$, *read* A, $\frac{A\,n-1}{B\,n-1}$.
45.	8.	*For* $\frac{n-6}{1}$, *read* $\frac{n-6}{2}$.

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